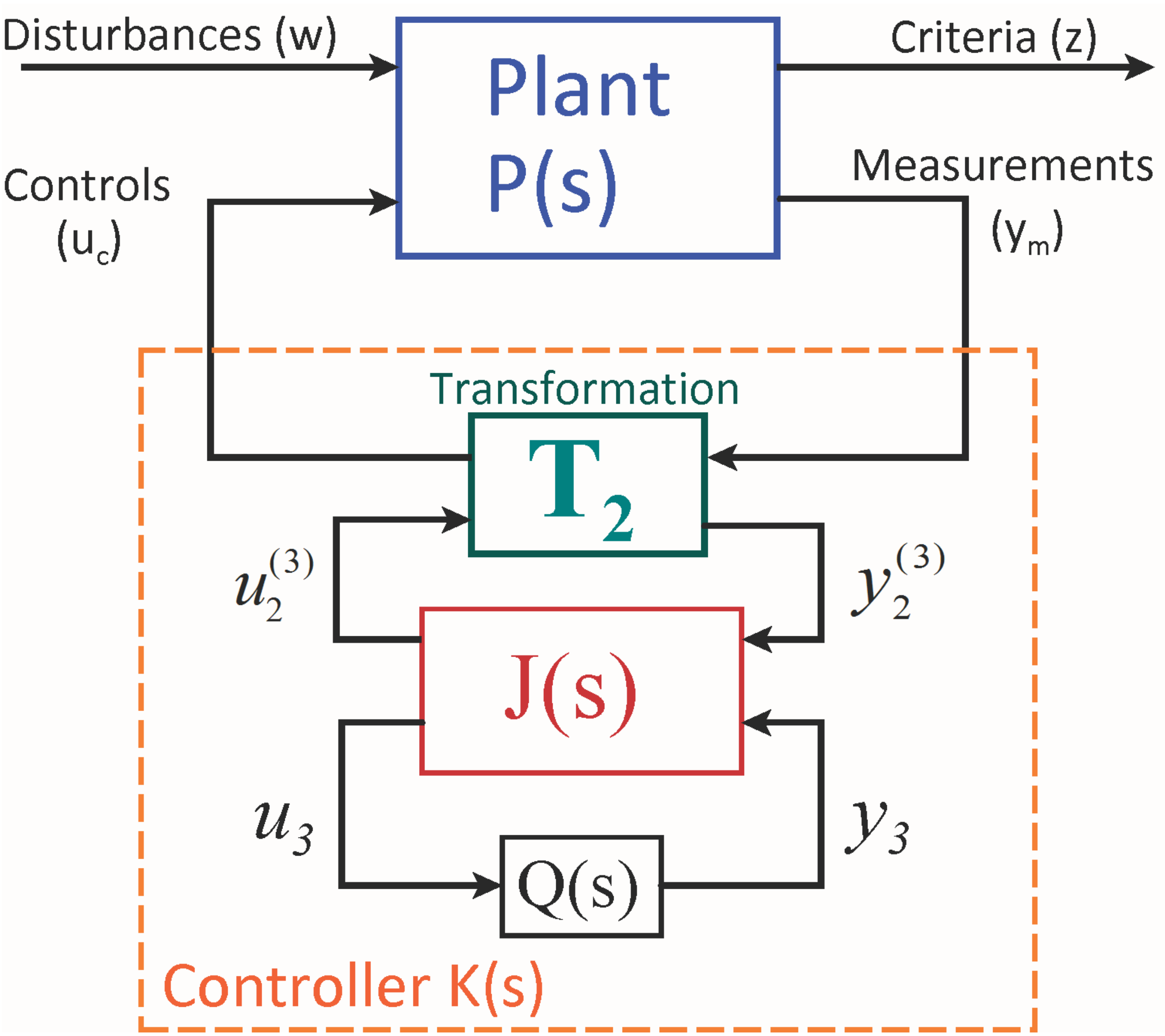
H-Infinity Control Design



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Background

The H_∞ algorithm is a powerful control synthesis method that attempts to minimize the infinity norm of the sensitivity transfer function matrix of the closed-loop system. The sensitivity function of a system is the transfer function between the disturbance inputs and some sensitive outputs that should be kept small, such as a spacecraft attitude or an airplane's angle of attack. The infinity norm (H_∞) is a measure of amplitude and it is the magnitude of the largest singular value over all frequencies. The H_∞ algorithm calculates a control system than minimizes the H_∞ of the system's sensitivity. The mathematical implementation of the H_∞ control synthesis algorithm requires two steps. The designer must first create a Synthesis Model (SM) that consists of 9 matrices including plant dynamics and performance requirements of the closed-loop system. The SM is then presented as input to the H_∞ algorithm that calculates the optimal control solution that will satisfy the design requirements. The algorithm requires the solution of two Riccati equations. In addition to the control requirements, some design intuition is needed in order to set up the synthesis model which is gained through experience.

In the design of a control system the engineer is faced with several requirements that must be satisfied with compromising solutions. The main goal is to design a control system that will provide good stability for the nominal plant. The control system must also behave properly by providing good performance with respect to commands and to external disturbances. That is, good response to commands and sufficient attenuation to disturbance signals. The control system must also be robust to unmodelled dynamics the uncertainty of which increases with frequency. It must, therefore, provide good attenuation at high frequencies in order to prevent uncertain plant dynamics (such as unmodelled flexible modes) from becoming unstable. In addition, the control system must be robust to small variations in plant parameters. In an aircraft, for example, the dynamic model is a function of several uncertain parameters such as the aerodynamic coefficients, the center of gravity, the center of pressure, dynamic pressure, altitude etc. In a spacecraft the uncertainties may be in the structural modes, the center of mass, the moments of inertia, etc. The SM includes all the closed-loop requirements of the system, such as: parameter uncertainties, unmodelled dynamics, environmental disturbance magnitudes, control limits, performance criteria magnitudes etc. The resulting H_∞ controller must satisfy or compromise those design requirements.

The H_∞ program included in Flixan not only solves the H_∞ optimization algorithm but it also includes a utility that helps the designer to create the SM interactively from the plant dynamics. The plant dynamics is a system that is usually created from the vehicle modeling program. The user separates the inputs into controls and disturbances, and the outputs into measurements and performance criteria. After separating them you end up with a SM consisting of 9 matrices that go into the H_∞ algorithm. The SM also includes some gains that define requirements on the disturbances and the performance criteria. They trade between control bandwidth, robustness to noise and un-modeled dynamics, and sensitivity. This documentation of the H_∞ control design program begins with a basic introduction of the H_∞ problem formulation. In chapter 2 we present the standard H_∞ SM and its mathematical solution. In chapter 3 we describe a more general SM formulation that includes direct transfers from inputs to outputs. In chapter 4 we demonstrate how to include parameter uncertainties in the plant using the Internal Feedback Loop (IFL) structure. In chapter 8 we describe the use of the H_∞ program. The references are in chapter 9, and in chapter 10 we demonstrate some control design examples.

1.0 Introduction to H_∞ Control System Design.

One of the great achievements in control theory during the late 1980's was the development of a systematic control design method that minimizes the infinity norm of the sensitivity function, i.e. the transfer function between the input disturbance inputs and the performance criteria outputs to be optimized. The advantage of this procedure is in characterizing the solution of the H_∞ problem in state-space form and solving it by means of two Riccati equations, a solution which is similar to the well-known LQG problem. Some of the contributors of this new theory are: J. Doyle, K. Glover, M. Safonov, P. Khargonekar, B.A. Francis, et. al. Consider the system in Figure 1.1 where G(s) is the plant model and K(s) is the controller, and let us assume that the plant has the State-Space representation of equation 1.1.

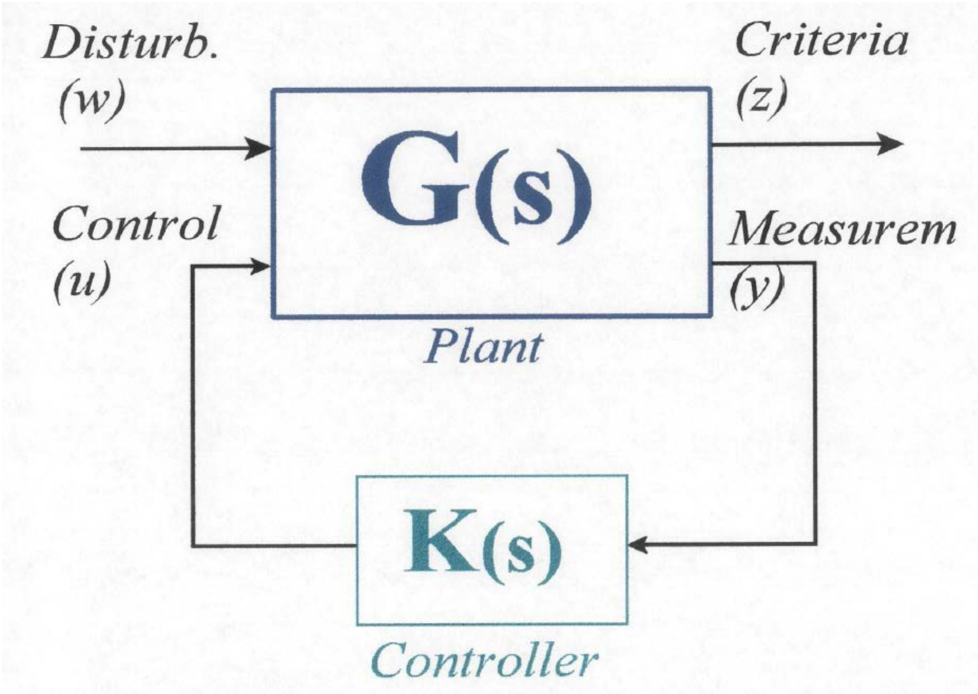


Figure 1.1 H-infinity controller minimizes system response between the disturbances (w) and the criteria (z)

$$\begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix} \tag{1.1}$$

Where:

- x is the plant state-vector of G(s) of dimension (n)
- u is the control inputs vector, consisting of (l) actuators
- y is the measurements vector (sensors) of dimension (m), measuring a linear combination of the plant's states plus noise.
- w is the external disturbances vector of dimension (l_w) greater than or equal to the number of measurements ($l_w \geq m$). The vector w consists of both input and output disturbances.
- z is the performance criterion vector of dimension (m_z). It is a set of variables consisting of a combination of states and control inputs that must be optimized by the algorithm, not necessarily actual outputs. It must be greater than or equal to the number of the control inputs, ($m_z \geq l$).

The H_∞ control problem can be described by the following statement. Find an admissible controller K(s) such that the infinity norm (i.e. the maximum singular value over the entire frequency range) of the transfer function from w to z in Figure 1.1, is less than a constant value (γ),

i.e. $\|T_{zw}\|_\infty < \gamma$ (1.2)

The first step in the H_∞ design process is to create a mathematical Synthesis Model that includes the basic plant dynamics, definition of disturbances and criteria to be optimized, and description of some uncertain internal plant parameters in order to improve the system's robustness to uncertainties. The H_∞ algorithm then reads the SM and calculates the control system K(s) that closes the loop between measurements and the controls. The success of the control design depends in the proper trade-off between performance of some outputs in response to commands, robustness against uncertainties and sensitivity of some variables to disturbance inputs. This can be adjusted by tweaking some scaling gains in the SM. The gains are always positive. There are input disturbance scaling gains which are initially set equal to the maximum magnitude of the corresponding disturbance input. There are also output criteria scaling gains, where initially each gain is set equal to the inverse of the maximum allowable magnitude of the corresponding output. This scaling normalizes the transfer function requirements between disturbances and criteria to be less than unity. The gains are adjusted after a few control design iterations to produce an acceptable trade-off between control bandwidth, robustness to uncertainties, system sensitivity to external disturbances and acceptable response to input commands. A simple simulation is used to examine the control system performance between gain adjustments.

The mathematical solution of the H_∞ optimization problem is similar to the LQG and it involves the solution of two Riccati equations: a state estimator and a state feedback problem. The two Riccati equations calculate the state feedback matrix F_∞ and the output injection matrix H_∞. The H_∞ algorithm solves either the state-feedback with estimator using both Riccati equations or only the state-feedback problem if the entire state vector is measurable. The control law is saved either as a state-space system or as a state-feedback matrix.

In section 2 we will describe the standard H_∞ state-feedback synthesis model and will present an algorithm that asymptotically minimizes the infinity norm of the sensitivity transfer function. The optimization algorithm requires some conditions to be satisfied by the SM which are described in Section 2.1. These conditions are not always satisfied by a general SM and a series of transformations of the original SM are applied in order to convert it to the standard model and satisfy the conditions. The transformations were derived by Safonov et al, in Ref.[2] and they are described in Section 2.2.2. The controller derived from the transformed SM must be back-transformed in order to match the original plant.

2.0 H_∞ Control via Full-State Feedback.

By full-state feedback we mean that the entire state vector is measurable and used for control. In this section we will describe the general formulation of the full state feedback synthesis model and present two solutions. The first solution is simple and it assumes that the matrix D_{11}=0. The second approach does not have the D_{11}=0 limitation, but it has a more complex solution consisting of three parts, (a) the solution of a simplified standard formulation, (b) the transformation of any general type of synthesis problem into the standard form, (c) the back-transformation of the controller obtained from the transformed system to match the original system.

The H_∞ control problem via full state feedback is formulated by the state-space synthesis model in equation 2.1. The vector $\underline{w}$ is the disturbance input of dimension (m_1), vector $\underline{u}$ is the control input of dimension (m_2), and vector $\underline{z}$ is the criterion output of dimension (p_1). The control design problem is to find a constant state-feedback matrix F_∞ that stabilizes the system, and minimizes the closed loop sensitivity transfer function between the disturbance $\underline{w}$ and the criterion vector $\underline{z}$, i.e. $\|T_{zw}\|_\infty < \gamma$.

$$\begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ I & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix} \qquad (2.1)$$

In Section 5 we present a method that transforms a robust control design problem including internal "structured" parameter variations into the formulation of Equation 2.1. This allows us to use full state-feedback H_∞ control design and derive controllers that reduce the system's sensitivity to internal parameter uncertainties, such as aero coefficients, etc.

2.1 Full State-Feedback Solution Assuming D_{11} is Zero

A simplified formulation for solving the full state feedback problem is to assume that the matrix $D_{11}=0$, and the following additional conditions must be satisfied:

(i) The pair (A, B_1) is stabilizable
(ii) The pair (C_1, A) is detectable
(iii) The pair (A, B_2) is stabilizable
(iv) $D^T_{12}C_1 = 0$
(v) $D^T_{12}D_{12} = I$ (2.1.1)

The matrix C_1 plays the role of penalizing the criteria outputs z̲. The matrix D_{12} penalizes the control inputs u̲, and it must be of full rank. If $D^T_{12}D_{12} \neq I$, but D_{12} is full rank we can scale the input by factoring D_{12} using singular value decomposition

$$D_{12} = U_1 \begin{bmatrix} 0 \\ \Sigma_1 \end{bmatrix} V_1^T \tag{2.1.2}$$

By inserting $V_1\, \Sigma^{-1}$ in series with the control signal (u), the new D_{12} matrix becomes

$$\hat{D}_{12} = D_{12} V_1 \Sigma_1^{-1} = U_1 \begin{bmatrix} 0 \\ I \end{bmatrix}; \qquad and \qquad \hat{D}_{12}^T \hat{D}_{12} = I \tag{2.1.3}$$

Define a matrix R as follows

$$R^{-1} = V_1 \Sigma_1^{-2} V_1^T \qquad then \qquad D_{12}^T D_{12} = R \tag{2.1.4}$$

The scaled H_∞ solution for the full state feedback is

$$X_\infty = Ric \begin{bmatrix} A & \left(\dfrac{B_1 B_1^T}{\gamma^2} - B_2 R^{-1} B_2^T \right) \\ -C_1^T C_1 & -A^T \end{bmatrix} \tag{2.1.5}$$

The state-feedback controller matrix F_∞ that satisfies $\|T_{zw}\|_\infty < \gamma$ is:

$$F_\infty = -R^{-1} B_2^T X_\infty \tag{2.1.6}$$

2.2 Full-State Feedback Solution for Non-Zero D_{11}

A more general solution of the full state feedback problem is presented here, assuming that matrix D_{11} is non zero. The solution is more complex and it consists of three parts:

(a) A solution based on the "Standard Synthesis Model", which assumes that the matrices D_{11} and D_{12} have certain structure
(b) The general design problem is transformed into the standard SM form, in order to apply the solution of the standard model and to obtain an H_∞ controller for the transformed SM, and
(c) The H_∞ controller is back-transformed using a reverse transformation to match the original model.

2.2.1 Full-State Feedback H_∞ Solution for the Standard Problem

Consider the SM formulation of Equation 2.1, and assume the following:

(i) The pair (A, B_2) is stabilizable.
(ii) $D_{12} = [0, I]^T$, and $D_{11}=0$

Define the following Hamiltonian matrix, solve a Riccati equation for X_∞, and obtain the state-feedback matrix F_∞ for the standard model.

$$X_\infty = Ric\begin{bmatrix} A - B_2 D_{12}^T C_1 & \dfrac{B_1 B_1^T}{\gamma^2} - B_2 B_2^T \\ -C_1^T\left(I - D_{12} D_{12}^T\right)C_1 & -\left(A - B_2 D_{12}^T C_1\right)^T \end{bmatrix}$$
$$F_\infty = -\left(B_2^T X_\infty + D_{12}^T C_1\right) \qquad (2.1.7)$$

There exists an internally stabilizing controller such that $\|T_{zw}\|_\infty < \gamma$ if and only if the following two conditions are satisfied:

(a) The Hamiltonian must have no pure imaginary eigenvalues, which means that X_∞ exists.
(b) The solution of the Riccati Equation, matrix X_∞ must be positive semidefinite, $X_\infty \geq 0$.

2.2.2 Synthesis Model Transformations

However, it is not always possible to satisfy the assumptions of the standard model: $D_{12}=[0, I]^T$ and $D_{11}=0$. We shall therefore present a procedure that is based on scaling and unimodular transformations by Safonov, to design a full state feedback gain matrix for the generic SM described in equation 2.1, and not limited by the above two assumptions. The generic SM is transformed to the standard form. The standard SM is used to design a preliminary H_∞ controller, and the controller is back-transformed to match the original plant.

Step-1, Scale Matrix D_{12}

First perform the singular value decomposition of the (p_1 x m_2) matrix D_{12} as shown in equation (2.2.1), where U_1 has (m_2) columns, and U_2 has (p_1-m_2) columns.

$$D_{12} = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} V^T \tag{2.2.1}$$

Then define new scaled input $u^{(1)}$ and output $z^{(1)}$ defined as follows:

$$u = S_u^{(1)} u^{(1)} = V \Sigma^{-1} u^{(1)}$$

$$z^{(1)} = S_z^{(1)} z = \begin{bmatrix} U_2^T \\ U_1^T \end{bmatrix} z \text{ , and } \ w^{(1)} = w \tag{2.2.2}$$

Substituting Equations 2.2.2 to Equations 2.1 we obtain the following modified state-space equations:

$$\begin{pmatrix} \dot{x} \\ z^{(1)} \end{pmatrix} = \begin{bmatrix} A^{(1)} & B_1^{(1)} & B_2^{(1)} \\ C_1^{(1)} & D_{11}^{(1)} & D_{12}^{(1)} \end{bmatrix} * \begin{bmatrix} x \\ w^{(1)} \\ u^{(1)} \end{bmatrix}$$

where:

$$A^{(1)} = A \qquad B_1^{(1)} = B_1 \qquad B_2^{(1)} = B_2 S_u^{(1)}$$

$$C_1^{(1)} = S_z^{(1)} C_1 \qquad D_{11}^{(1)} = S_z^{(1)} D_{11} \qquad D_{12}^{(1)} = \begin{bmatrix} 0 \\ I \end{bmatrix} \tag{2.2.3}$$

Step-2, Scale the (p_1 x m_1) Matrix D_{11}

Using the following unimodular transformation

$$\begin{pmatrix} z^{(1)} \\ w^{(1)} \end{pmatrix} = \begin{bmatrix} X_1^{-1/2} & X_1^{-1/2} D_{11}^{(1)} \\ \left(D_{11}^{(1)}\right)^T X_1^{-1/2} & X_2^{-1/2} \end{bmatrix} \begin{pmatrix} z^{(2)} \\ w^{(2)} \end{pmatrix} \quad \textit{where}$$

$$X_1 = I - D_{11}^{(1)} \left(D_{11}^{(1)}\right)^T \quad \textit{and} \quad X_2 = I - \left(D_{11}^{(1)}\right)^T D_{11}^{(1)}$$

Assuming that $u^{(2)} = u^{(1)}$ we obtain the following set of state-space equations where the matrix D_{11} is now zero.

$$\begin{pmatrix} \dot{x} \\ z^{(2)} \end{pmatrix} = \begin{bmatrix} A^{(2)} & B_1^{(2)} & B_2^{(2)} \\ C_1^{(2)} & D_{11}^{(2)} & D_{12}^{(2)} \end{bmatrix} * \begin{bmatrix} x \\ w^{(2)} \\ u^{(2)} \end{bmatrix}$$

where:

$$A^{(2)} = A^{(1)} + B_1^{(1)} \left(D_{11}^{(1)}\right)^T X_1^{-1} C_1^{(1)} \qquad B_1^{(2)} = B_1^{(1)} X_2^{-1/2} \qquad D_{11}^{(2)} = 0$$

$$B_2^{(2)} = B_2^{(1)} + B_1^{(1)} \left(D_{11}^{(1)}\right)^T X_1^{-1} D_{12}^{(1)} \qquad C_1^{(1)} = X_1^{-1/2} C_1^{(1)} \qquad D_{12}^{(2)} = X_1^{-1/2} D_{12}^{(1)}$$

Step-3, Repeat Step-1 and Rescale Matrix $D_{12}^{(2)}$

Perform the singular value decomposition of the (p_1 x m_2) matrix $D_{12}^{(2)}$ as in step-1

$$D_{12}^{(2)} = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} V^T \qquad (2.2.9)$$

Where: U_1 has (m_2) columns, and U_2 has (p_1-m_2) columns. Notice, the matrices U_1, U_2, Σ, and V are different from those in step-1.

Then define new scaled input $u^{(3)}$ and scaled output $z^{(3)}$ defined as follows:

$$u^{(2)} = S_u^{(3)} u^{(3)} = V \Sigma^{-1} u^{(3)}$$

$$z^{(3)} = S_z^{(3)} z^{(2)} = \begin{bmatrix} U_2^T \\ U_1^T \end{bmatrix} z^{(2)} \text{ , } \textit{and} \;\; w^{(3)} = w^{(2)} \qquad (2.2.10)$$

Substituting Equations 2.2.10 to Equations 2.2.7 we obtain the following synthesis model modifications.

$$\begin{pmatrix} \dot{x} \\ z^{(3)} \end{pmatrix} = \begin{bmatrix} A^{(3)} & B_1^{(3)} & B_2^{(3)} \\ C_1^{(3)} & D_{11}^{(3)} & D_{12}^{(3)} \end{bmatrix} * \begin{bmatrix} x \\ w^{(3)} \\ u^{(3)} \end{bmatrix}$$

where:

$$A^{(3)} = A^{(2)} \qquad B_1^{(3)} = B_1^{(2)} \qquad B_2^{(3)} = B_2^{(2)} S_u^{(3)}$$

$$C_1^{(3)} = S_z^{(3)} C_1^{(2)} \qquad D_{11}^{(3)} = S_z^{(3)} D_{11}^{(2)} = 0 \qquad D_{12}^{(3)} = \begin{bmatrix} 0 \\ I \end{bmatrix} \tag{2.2.12}$$

Step-4: Determine the State Feedback Matrix F_∞

The transformed system, described in Equations 2.2.12, is now in the standard form that satisfies conditions (i) and (ii) in section (2.2.1), and the standard solution of Equations 2.1.7 can be used to calculate the state feedback gain matrix $F_\infty^{(3)}$ for the transformed system. The state feedback gain matrix F_∞ for the original system of Equations (2.1) can be obtained by back-transforming the controller as follows:

$$F_\infty = S_u^{(1)} S_u^{(3)} F_\infty^{(3)} \text{ since: } u = S_u^{(1)} S_u^{(3)} u^{(3)} = S_u^{(1)} S_u^{(3)} F_\infty^{(3)} x \tag{2.2.13}$$

3.0 H∞ Control Design Using Output Feedback

In Section 2 we assumed that the plant's state vector x is directly measurable and the resulting controller is not dynamic but a state-feedback gain matrix. We will now formulate the output feedback H_∞ problem. In this case the input to the control system comes from the plant's output measurements (y_m) which is a linear combination of the states. A state estimator is therefore needed to estimate the state vector from the measurements. The controller is a system consisting of a state estimator and a state-feedback gain matrix. They are combined together in a dynamic multivariable controller. The design of the H_∞ controller/ estimator is defined by the Synthesis Model which in general it includes plant dynamics, parameter uncertainties, disturbance inputs, criteria outputs, and gains that trad performance against robustness.

3.1 H∞ Synthesis Model (SM)

The original quadruple matrix system of a vehicle alone that consists of inputs, states and outputs is not sufficient for H∞ design. The H∞ method requires more variables and parameters in order to refine the optimization. The SM is a state-space system consisting of 9 matrices which captures the plant dynamics and the control system performance requirements that will be traded by the optimization algorithm. The SM in general includes: control inputs (u_c), commands to regulated outputs, exogenous disturbance inputs (w), measurements (y_m), measurement noise, and criteria outputs to be optimized (z). It may also include parameter uncertainties which are described with additional input-output pairs (w_p, z_p). The disturbance inputs and the criteria outputs are scaled by gains which affect the control system bandwidth. Equation 3.1 shows a typical synthesis model representation for output feedback control.

$$\begin{bmatrix} \dot{x} \\ z \\ y_m \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} x \\ w \\ u_c \end{bmatrix} \tag{3.1}$$

Where:

$\underline{x}$	is the (n) state-vector of the design plant
$\underline{u}_c$	is the control input vector, size (l)
y_m	is the measurements vector of size (m)
$\underline{w}$	is the disturbances vector at the plant input and output $w=[w_i, w_o]$
z	is the criterion vector including criteria on the control inputs $z=[z_o, z_i]$

The disturbance input (w) consists of disturbances at the plant input and also at the measurements. It may also include commands of regulated outputs, and inputs from structured uncertainties. It includes two parts: (a) an exogenous input disturbance vector $\underline{w}_i$ of dimension (l_w) that excites the states, and (b) an output disturbance $\underline{w}_o$ of dimension (m) that represents sensor noise or used to model uncertainty in the measurement. It defines how disturbances enter the plant at the inputs and at the sensors.

Similarly, the criterion output vector (z) consists of variables that must be optimized by the H∞ algorithm. It also consists of two parts: (a) the criterion $\underline{z}_o$ of dimension (m_z) that is a linear combination of the states, and (b) the criterion (z_i) of dimension (l) that penalizes the control inputs (u_c). The structured uncertainties create additional fictitious inputs in (w_i) and in the outputs (z_o). One input/ output pair per uncertainty.

Some of the matrices in the SM are extracted from the plant dynamics, while others are design parameters which are defined by the control designer. The matrices that are part of the plant dynamics are: the state-transition matrix A, the control input matrix B_2, the input disturbance matrix B_1, and the measurements output matrix C_2. The matrices C_1, D_{12}, and D_{21} are not necessarily part of the physical plant. They are design parameters used to trade system robustness versus performance. Matrix C_1 defines a set of criteria variables in the H_∞ optimization. Matrix D_{12} penalizes the control inputs and it is used to adjust the closed loop system bandwidth. Matrix D_{21} defines the effects of measurement noise, uncertain dynamics, or sensor noise in the measurements. The size of the matrix D12 is (lxl), and matrix D21 is (mxm). They must be square, full rank and diagonal for meaningful results.

3.2 Standard Output Feedback H_∞ Synthesis Model

Equation 3.1 represents the formulation of a generic SM and the H_∞ algorithm is expected to minimize the infinity norm of the sensitivity transfer function between the disturbance inputs (w) and the output criteria (z). However, this formulation is not easily solved directly. We have a solution for the standard H_∞ SM and in order to solve equations 3.1 we must first transform it to the "Standard Form" for which there is a simple H_∞ solution, and then back-transform the controller. The "Standard” H_∞ solution requires the following 5 conditions to be satisfied by the SM.

(i) The pairs (A, B_1) and (A, B_2) be stabilizable.
(ii) The pairs (C_1, A) and (C_2, A) be detectable.
(iii) The matrix product $(D_{12}^T D_{12})= I_l$
(iv) The matrix product $(D_{21} D_{21}^T)= I_m$
(v) The matrices $D_{11}=0$ and $D_{22}=0$

Condition (iii) implies that the number of output criteria (z) must be greater than or equal to the number (l) of controls u_c. Condition (iv) implies that the number of disturbances (w) must be greater than or equal to the number (m) of the measurements y_m. The conditions (iii) to (v), however, are not easily satisfied without a transformation on the SM. This transformation, by Safonov in Ref[4], is performed by a series of scaling and loop shifting operations that transform the generic H_∞ SM structure to the standard form of equations 3.1. The resulting controller must be back-transformed in order to be used with the original plant. In Section 3.3 we present the mathematical solution for the Standard H-infinity Problem.

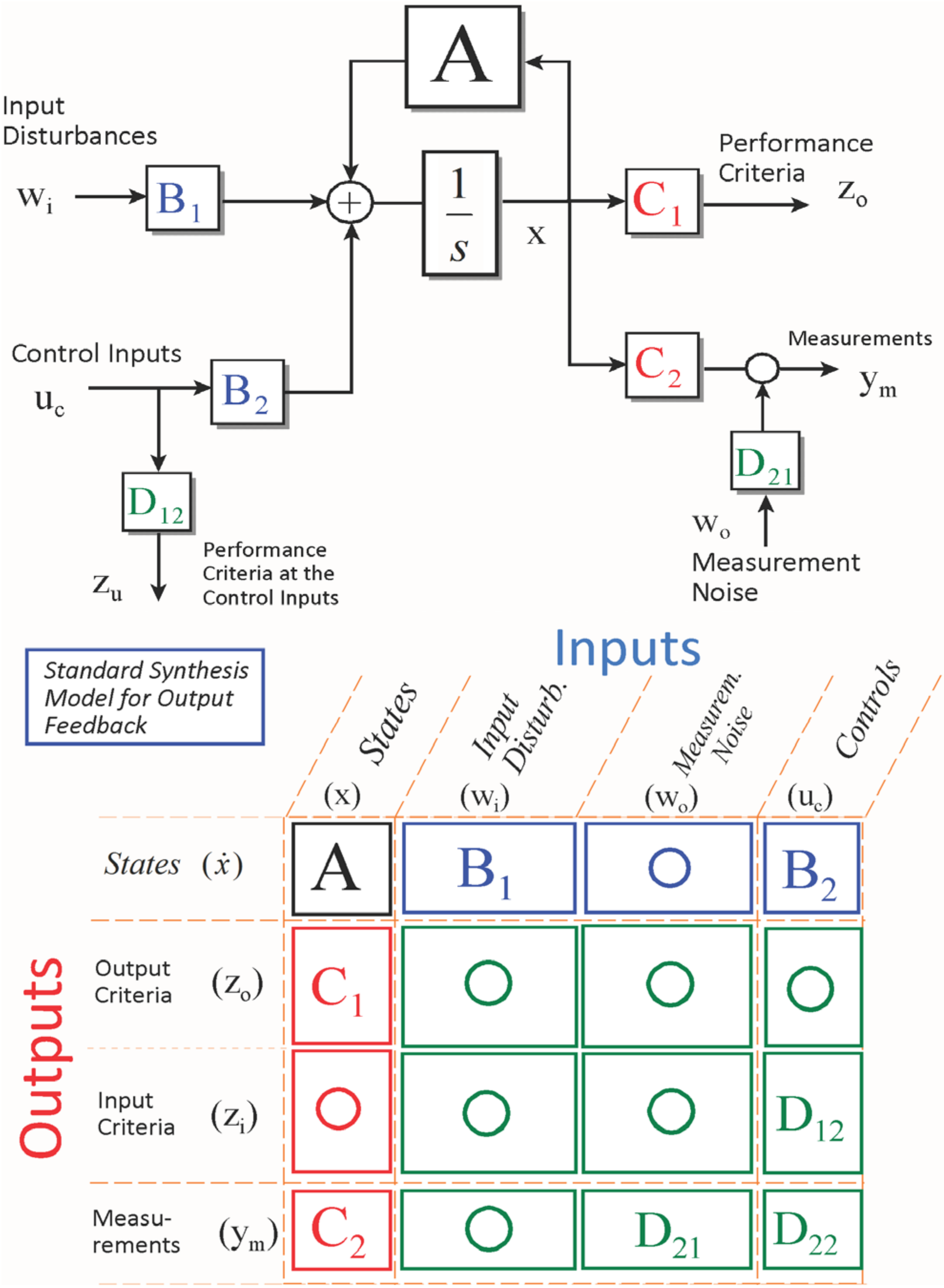


Figure 3.1 Standard Synthesis Model for Output Feedback H-Infinity Control Design

3.3 Solution for the Standard H∞ Output Feedback Problem

This H∞ solution requires a standard Synthesis Model of the form shown in Figure 3.1 and satisfies the conditions in Section 3.2 where the matrices D_{11} and D_{22} are zero and the matrices ($D_{12}^T D_{12}$) and ($D_{21} D_{21}^T$) are the identity matrices. The algorithm executes the following steps:

Step-1: Define a Hamiltonian matrix for the state-feedback controller, solve a Riccati equation for X_∞ and obtain the state-feedback matrix F_∞, as follows:

$$X_\infty = Ric\begin{bmatrix} A - B_2 D_{12}^T C_1 & \dfrac{B_1 B_1^T}{\gamma^2} - B_2 B_2^T \\ -\tilde{C}_1^T \tilde{C}_1 & -\left(A - B_2 D_{12}^T C_1\right)^T \end{bmatrix} \quad where$$

$$\tilde{C}_1 = \left(I - D_{12} D_{12}^T\right) C_1 \quad and \quad F_\infty = -\left(B_2^T X_\infty + D_{12}^T C_1\right)$$

The following two conditions must be satisfied:

1) The Hamiltonian must have no pure imaginary eigenvalues, which means that X_∞ exists.
2) The solution of the Riccati Equation, matrix X_∞ must be positive semidefinite, $X_\infty > 0$.

Step-2: Define a second Hamiltonian matrix for the estimator, solve a Riccati equation for Y_∞, and obtain the output injection matrix H_∞

$$Y_\infty = Ric\begin{bmatrix} \left(A - B_1 D_{21}^T C_2\right)^T & \dfrac{C_1^T C_1}{\gamma^2} - C_2^T C_2 \\ -\tilde{B}_1 \tilde{B}_1^T & -\left(A - B_1 D_{21}^T C_2\right) \end{bmatrix} \quad where$$

$$\tilde{B}_1 = B_1\left(I - D_{21}^T D_{21}\right); \quad H_\infty = -\left(Y_\infty C_2^T + B_1 D_{21}^T\right);$$

$$Z = \left(I - \gamma^{-2} Y_\infty X_\infty\right)^{-1}$$

The following two conditions must be satisfied:

1) The Hamiltonian must have no pure imaginary eigenvalues, which means that Y_∞ exists.
2) The solution of the Riccati Equation matrix Y_∞ must be positive semidefinite, $Y_\infty > 0$.

Step-3: Calculate the spectral radius ρ, which is the magnitude of the largest eigenvalue of the product ($X_\infty Y_\infty$). We must now find the smallest value of a parameter **γ** that satisfies the condition: $\|T_{zw}\|_\infty < \gamma$; where T_{zw} is the transfer function between the disturbances (w) to the criteria (z). We iterate the parameter **γ** beginning with large values and try to satisfy the three conditions below. We gradually reduce γ until one of the three conditions below is violated, in which case we accept the smallest that doesn't violate the 3 conditions.

a) $\rho(X_\infty Y_\infty) < \gamma^2$ (3.3.3)
b) The matrices X_∞ and Y_∞ from the Riccati Equations must be positive semi-definite, i.e. $X_\infty \geq 0$ and $Y_\infty \geq 0$.
c) The Hamiltonian matrices X_∞ and Y_∞ must have no imaginary eigenvalues.

If the above conditions are satisfied by a value of γ, then we can assume that the condition $\|T_{zw}\|_\infty < \gamma$ will also be satisfied. If one of the above conditions is violated, we must increase γ and repeat the procedure.

Step-4: When **γ** is large enough to satisfy the above 3 conditions and yet small enough to produce a satisfactory sensitivity $\|T_{zw}\|_\infty$ then we can calculate the following matrix Z_∞.

$$Z_\infty = \left(I - \gamma^{-2} Y_\infty X_\infty\right)^{-1} \quad (3.3.4)$$

There is a family of stabilizing controllers that satisfy the sensitivity condition for "nominal performance" $\|T_{zw}\|_\infty < \gamma$. The general form of the controller K(s) can be expressed in state-space form, as shown in Equation 3.3.5 and Figure 3.3.6. The controller consists of two parts, and it can be written as K(s)= F_l(J,Q). Q(s) can be any given stable transfer function that satisfies the condition $\|Q\|_\infty < \gamma$. J(s) is a two-vector input, two vector output transfer function matrix. The presence of Q(s) in the controller is optional. In fact, Q(s)=0 may be sufficient. Q(s) is defined by the following matrix equation

$$y_l(s) = Q(s) u_m(s)$$

Where: the dimension of vector u_m is equal to the number of measurements (m), and the dimension of vector y_l is equal to the number of the control inputs (l). Note that as γ increases to infinity the solution of the H∞ control problem becomes identical to the LQG problem.

The H∞ controller solution that stabilizes the standard synthesis model of Figure 3.1 and satisfies the specified sensitivity requirements is described by the state-space equations 3.3.5. All of the parameters are derived from the SM.

$$\begin{bmatrix} \dot{x}_c \\ u_2^{(3)} \\ u_3 \end{bmatrix} = \begin{bmatrix} A_C & -Z_\infty H_\infty & Z_\infty B_3 \\ F_\infty & 0 & I \\ -C_3 & I & 0 \end{bmatrix} \begin{bmatrix} x_c \\ y_2^{(3)} \\ y_3 \end{bmatrix}$$

$$\text{where} \quad A_C = A + B_2 F + \gamma^{-2} B_1 B_1^T X_\infty + Z_\infty H_\infty \left(C_2 + \gamma^{-2} D_{21} B_1^T X_\infty \right)$$

$$B_3 = B_2 + \gamma^{-2} Y_\infty C_1^T D_{12} \qquad C_3 = C_2 + \gamma^{-2} D_{21} B_1^T X_\infty$$

$$Z_\infty = \left(I - \gamma^{-2} Y_\infty X_\infty \right)^{-1}$$

Equation 3.3.5 H∞ Controller for the Plant Transformed into Standard Form

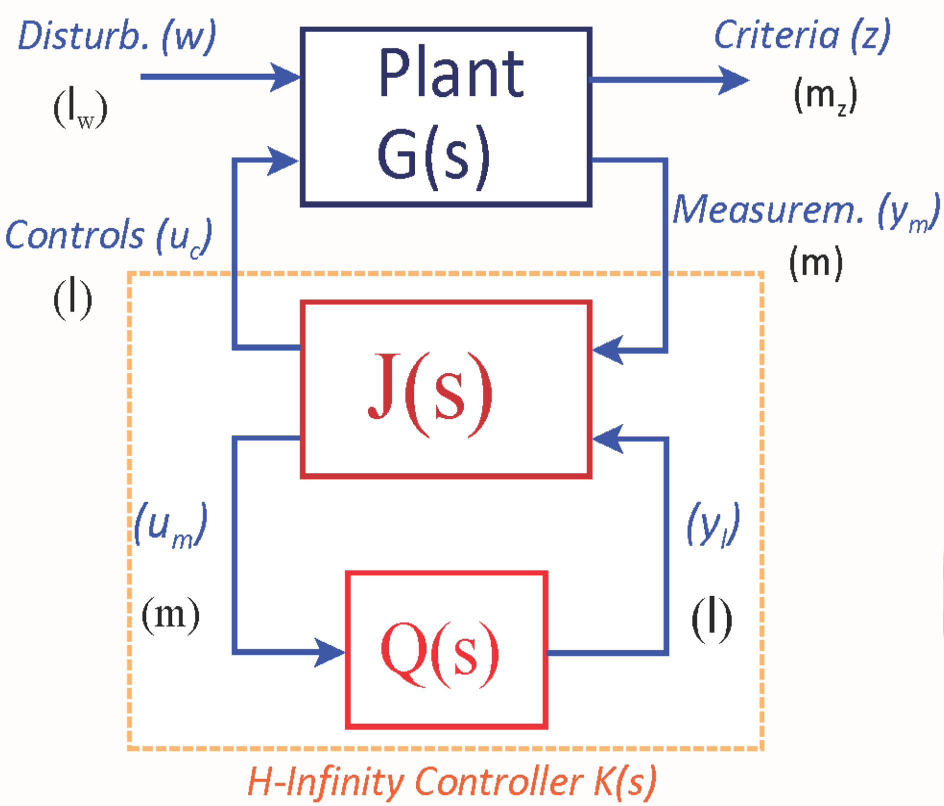
The actuator input command is $u_2 = F_\infty \hat{x}$, where $\hat{x}$ is the estimated state vector. The estimator dynamics when the lower feedback loop Q(s)=0, is

$$\dot{\hat{x}} = \left(A + \gamma^{-2} B_1 B_1^T X_\infty \right) \hat{x} + B_2 u_2 - Z_\infty H_\infty \left(y_2 - \hat{y}_2 \right)$$

Where: $\hat{y}_2$ is the estimate of the output measurement, $\hat{y}_2 = \left(C_2 + \gamma^{-2} D_{21} B_1^T X_\infty \right) \hat{x}$

The H_∞ Controller for the Standard Problem is:

$$J(s)=\begin{bmatrix} A+B_2F_\infty+\gamma^{-2}B_1B_1^TX_\infty+Z_\infty H_\infty\left(C_2+\gamma^{-2}D_{21}B_1^TX_\infty\right) & -Z_\infty H_\infty & Z_\infty\left(B_2+\gamma^{-2}Y_\infty C_1^TD_{12}\right) \\ F_\infty & 0 & I \\ -\left(C_2+\gamma^{-2}D_{21}B_1^TX_\infty\right) & I & 0 \end{bmatrix}$$



The Controller K(s) is:

$$\begin{pmatrix} u_c \\ u_m \end{pmatrix} = \begin{bmatrix} J_{11}(s) & J_{12}(s) \\ J_{21}(s) & J_{22}(s) \end{bmatrix}\begin{pmatrix} y_m \\ y_l \end{pmatrix}$$

where: $y_l = Q(s)u_m$

3.4 General Formulation of the Synthesis Model

The simple solution of the Standard H-infinity SM formulation presented in Section 3.3 requires the conditions (iii-v) of Section 3.2 to be satisfied in order to use the controller. In general, the SM may have non-zero direct transfer matrices D11, D22, and not satisfy the conditions (iii) and (iv). The solution in this case is more complex and the SM requires a series of transformations in order to be transformed and comply with the standard form of Figure 3.1. These transformations will be described in Section 3.5. In this section we present a more general synthesis model that is typically obtained when setting up an H_∞ problem. This model includes the plant dynamics, the control inputs, disturbance and command inputs, the measurements, and criteria outputs that must be minimized. The noticeable differences between the standard and the generic SM is that we now have direct transfer from the exogenous inputs w_i to the output criterion and measurement vectors (z_o and y_m) via the matrices D_{1111} and D_{21w} respectively. The exogenous inputs are either disturbances, coupling directly to the plant input via matrix B_1, or commands that go directly to the regulated outputs in vector z_o via D_{1111}, due to the fact that some of the criteria vector elements z_o consist of commands minus system responses. In addition, there is also a direct transfer from the control u_c to the output criteria z_o and to measurements vector y_m via the matrices D_{12u} and D_{22} respectively, see Figure 3.4.

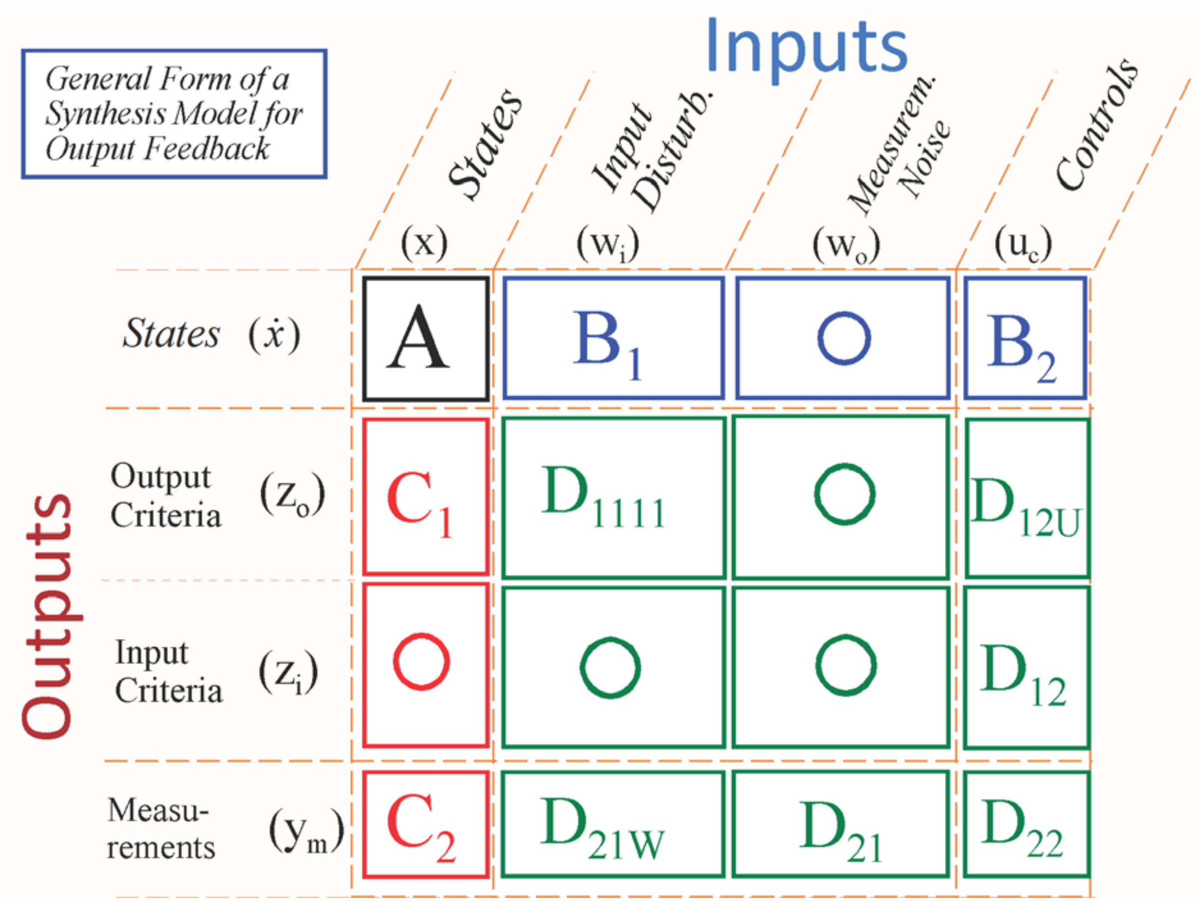


Figure 3.4 State-Space Formulation of the General H_∞ Synthesis Model

3.5 Transformation of the General H∞ SM Using Scaling and Loop-Shifting Operations

The specific H∞ synthesis procedure described in Section 3.3 assumes that the conditions (iii to v) stated in Section 3.2 are satisfied. These conditions, however, greatly reduce the applicability of the design algorithm. In this section we will present a series of operations that can be applied to the generic SM of Figure 3.4 which doesn't meet the conditions of the standard SM in order to transform it to the specific form. For a given general D matrix and a desired upper H∞ bound γ, the following series of scaling and loop shifting operations will transform the system to the required standard form. Knowledge of γ will be required in order to zero out the D_{11} matrix.

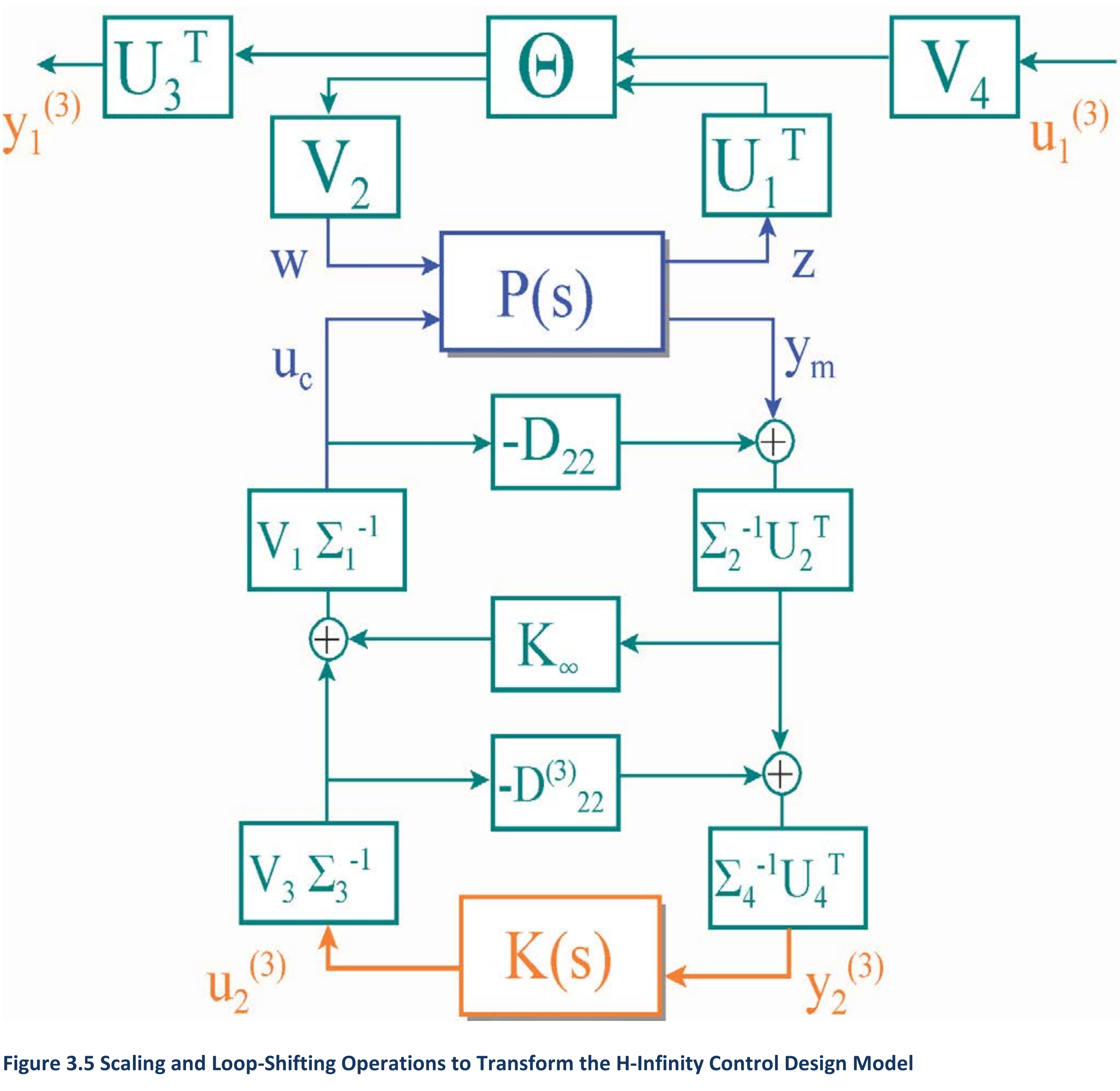


Figure 3.5 Scaling and Loop-Shifting Operations to Transform the H-Infinity Control Design Model

These transformations must be repeated each time γ is changed as one iterates on γ to approach the optimal H_∞ solution. Most of these transformations are norm-preserving, and they do not change the system's H-infinity norm. The transformation Θ however, that zeros out matrix D_{11}, does not preserve the system's H-infinity norm but only its upper bound γ. In other words, although the transformed system's H-infinity norm may differ from that of the original system, the transformed system's H-infinity norm will be less than γ, if and only if the original system's H-infinity norm is less than γ. The transformations were derived by Safonov et al. in reference [2]. The transformation algorithm is described by the following steps:

Step-1: Use Singular Value Decomposition to factor the matrices D_{12} and D_{21} and perform the first set of transformations of the original SM matrices, as shown below:

$$D_{12} = U_1 \begin{bmatrix} 0 \\ \Sigma_1 \end{bmatrix} V_1^T \qquad D_{21} = U_2 \begin{bmatrix} 0 & \Sigma_2 \end{bmatrix} V_2^T$$

$$B_1^{(2)} = B_1 V_2 \qquad B_2^{(2)} = B_2 V_1 S_1^{-1} \qquad D_{11}^{(2)} = U_1^T D_{11} V_2 \qquad D_{12}^{(2)} = U_1^T D_{12} V_1 S_1^{-1}$$

$$C_1^{(2)} = U_1^T C_1 \qquad C_2^{(2)} = S_2^{-1} U_2^T C_2 \qquad D_{21}^{(2)} = S_2^{-1} U_2^T D_{21} V_2 \qquad D_{22}^{(2)} = S_2^{-1} U_2^T D_{22} V_1 S_1^{-1}$$

Step-2: Scale and partition the new matrix D_{11} into a (2x2) block matrix, where the lower right block D_{1122} has the same dimension as D_{22}^T, and define the following matrix K_∞

$$D_{11}^{(2)} = U_1^T D_{11} V_2 = \begin{bmatrix} D_{1111} & D_{1112} \\ D_{1121} & D_{1122} \end{bmatrix}$$

$$K_\infty = -\left(D_{1122} + D_{1121}(\gamma^2 I - D_{1111}^T D_{1111})^{-1} D_{1111}^T D_{1112}\right)$$

Step-3: Let the matrix **M** and the transformation matrix **Θ** in Figure 3.5.1 be defined as follows:

$$M = \begin{bmatrix} D_{1111} & D_{1112} \\ D_{1121} & D_{1122} + K_\infty \end{bmatrix} \quad \textit{and}$$

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} = \begin{bmatrix} -M & \left(I - \gamma^{-2} M M^T\right)^{1/2} \\ \left(I - \gamma^{-2} M^T M\right)^{1/2} & \gamma^{-2} M^T \end{bmatrix}$$

At this point the transformed system's D_{11} matrix is zero. The other matrices are:

$$A^{(3)} = A + B_1^{(2)} M^T \left(I - MM^T\right)^{-1} C_1^{(2)} \qquad B_1^{(3)} = B_1^{(2)} \Theta_{21}^{-1} \qquad C_1^{(3)} = \Theta_{12}^{-1} C_1^{(2)}$$

$$B_2^{(3)} = B_2^{(2)} + B_1^{(2)} M^T \left(I - MM^T\right)^{-1} D_{12}^{(2)} \qquad C_2^{(3)} = C_2^{(2)} + D_{21}^{(2)} M^T \left(I - MM^T\right)^{-1} C_1^{(2)}$$

$$D_{21}^{(3)} = D_{21}^{(2)} \Theta_{21}^{-1} \qquad D_{12}^{(3)} = \Theta_{12}^{-1} D_{12}^{(2)}$$

$$D_{11}^{(3)} = \Theta_{12}^{-1}\left(M \Theta_{21}^{-1} - \Theta_{12}^{-1} M\right) = 0 \qquad D_{22}^{(3)} = D_{21}^{(2)} M^T \left(I - MM^T\right)^{-1} D_{12}^{(2)}$$

Step-4: Use Singular Value Decomposition to factor the matrices D_{12} and D_{21} that were generated in step-3. This final transformation will bring the synthesis model to the standard form. At this point all the blocks appearing in figure 3.5 have been calculated. The transformed synthesis model is the transfer function from: $u^{(3)}{}_1$ and $u^{(3)}{}_2$, to: $y^{(3)}{}_1$ and $y^{(3)}{}_2$. It consists of the following matrices that have been transformed as shown:

$$D_{12}^{(3)} = U_3 \begin{bmatrix} 0 \\ \Sigma_3 \end{bmatrix} V_3^T \qquad and \qquad D_{21}^{(3)} = U_4 \begin{bmatrix} 0 & \Sigma_4 \end{bmatrix} V_4^T$$

$$B_1^{(4)} = B_1^{(3)} V_4 \qquad B_2^{(4)} = B_2^{(3)} V_3 S_3^{-1} \qquad D_{12}^{(4)} = U_3^T D_{12}^{(3)} V_3 S_3^{-1} = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

$$C_1^{(4)} = U_3^T C_1^{(3)} \qquad C_2^{(4)} = S_4^{-1} U_4^T C_2^{(3)} \qquad D_{21}^{(4)} = S_4^{-1} U_4^T D_{21}^{(3)} V_4 = \begin{bmatrix} 0 & I \end{bmatrix}$$

$$D_{22}^{(4)} = S_4^{-1} U_4^T D_{22}^{(3)} V_3 S_3^{-1} = 0 \qquad D_{11}^{(4)} = U_3^T D_{11}^{(3)} V_4 = 0$$

The transformations above and below the plant P can be grouped together into two matrices T_1 and T_2, respectively, where the transformation T_1 above the plant P(s) is:

$$\begin{bmatrix} y_1^{(3)} \\ w \end{bmatrix} = \begin{bmatrix} T_{111} & T_{112} \\ T_{121} & T_{122} \end{bmatrix} \begin{pmatrix} u_1^{(3)} \\ z \end{pmatrix} \quad where$$

$$T_{111} = U_3^T \Theta_{11} V_4 \qquad T_{112} = U_3^T \Theta_{12} U_1^T \qquad T_{121} = V_2 \Theta_{21} V_4 \qquad T_{122} = V_2 \Theta_{22} U_1^T$$

The transformation T_2 below the plant P(s) is:

$$\begin{bmatrix} u_c \\ y_2^{(3)} \end{bmatrix} = \begin{bmatrix} T_{211} & T_{212} \\ T_{221} & T_{222} \end{bmatrix} \begin{pmatrix} y_m \\ u_2^{(3)} \end{pmatrix} \quad where$$

$$T_{211} = V_1 \Sigma_1^{-1} K_\infty \Sigma_2^{-1} U_2^T \left(I - L_1\right)^{-1} \qquad T_{212} = V_1 \Sigma_1^{-1} \left(I - L_2\right)^{-1} V_3 \Sigma_3^{-1}$$

$$T_{221} = \Sigma_4^{-1} U_4^T \Sigma_2^{-1} U_2^T \left(I - L_1\right)^{-1}$$

$$T_{222} = -\Sigma_4^{-1} U_4^T \left[\Sigma_2^{-1} U_2^T D_{22} V_1 \Sigma_1^{-1} \left(I - L_2\right)^{-1} - D_{22}^{(3)} \right] V_3 \Sigma_3^{-1}$$

$$L_1 = -D_{22} V_1 \Sigma_1^{-1} K_\infty \Sigma_2^{-1} U_2^T \qquad L_2 = -K_\infty \Sigma_2^{-1} U_2^T D_{22} V_1 \Sigma_1^{-1}$$

The D matrix of the transformed system now has components $D_{11}=0$, $D_{22}=0$, $D_{12}=[0, I]^T$, and D21=[0, I]. Since all the conditions are satisfied a controller can be calculated that minimizes the H_∞ norm of the sensitivity transfer function for the transformed model. The H_∞ controller that stabilizes the transformed plant P'(s) is shown in Equation 3.5.1.

3.6 Modified H_∞ Controller for the Original Plant

In Figure 3.6 the controller J(s) is for the modified SM which was transformed into the standard form and was obtained from the standard H_∞ algorithm described in Section 3.3. This controller must now be back-transformed using the T_2 transformation matrix in order to be applied on the original plant P(s) and have the same effect. The state-space representation of the controller for the original plant is shown in Equation 3.5.1. The matrices Ac, B_3 and C_3 were defined in equation 3.3.5.

$$\begin{bmatrix} \dot{x}_c \\ u_c \\ u_3 \end{bmatrix} = \begin{bmatrix} A_C - Z_\infty H_\infty T_{222} F_\infty & -Z_\infty H_\infty T_{221} & Z_\infty \left(B_3 - H_\infty T_{222}\right) \\ T_{212} F_\infty & T_{211} & T_{212} \\ -\left(C_3 - T_{222} F_\infty\right) & T_{221} & T_{222} \end{bmatrix} \begin{bmatrix} x_c \\ y_m \\ y_3 \end{bmatrix}$$

Equation 3.5.1 Transformed H_∞ Controller for the Original Generic Plant

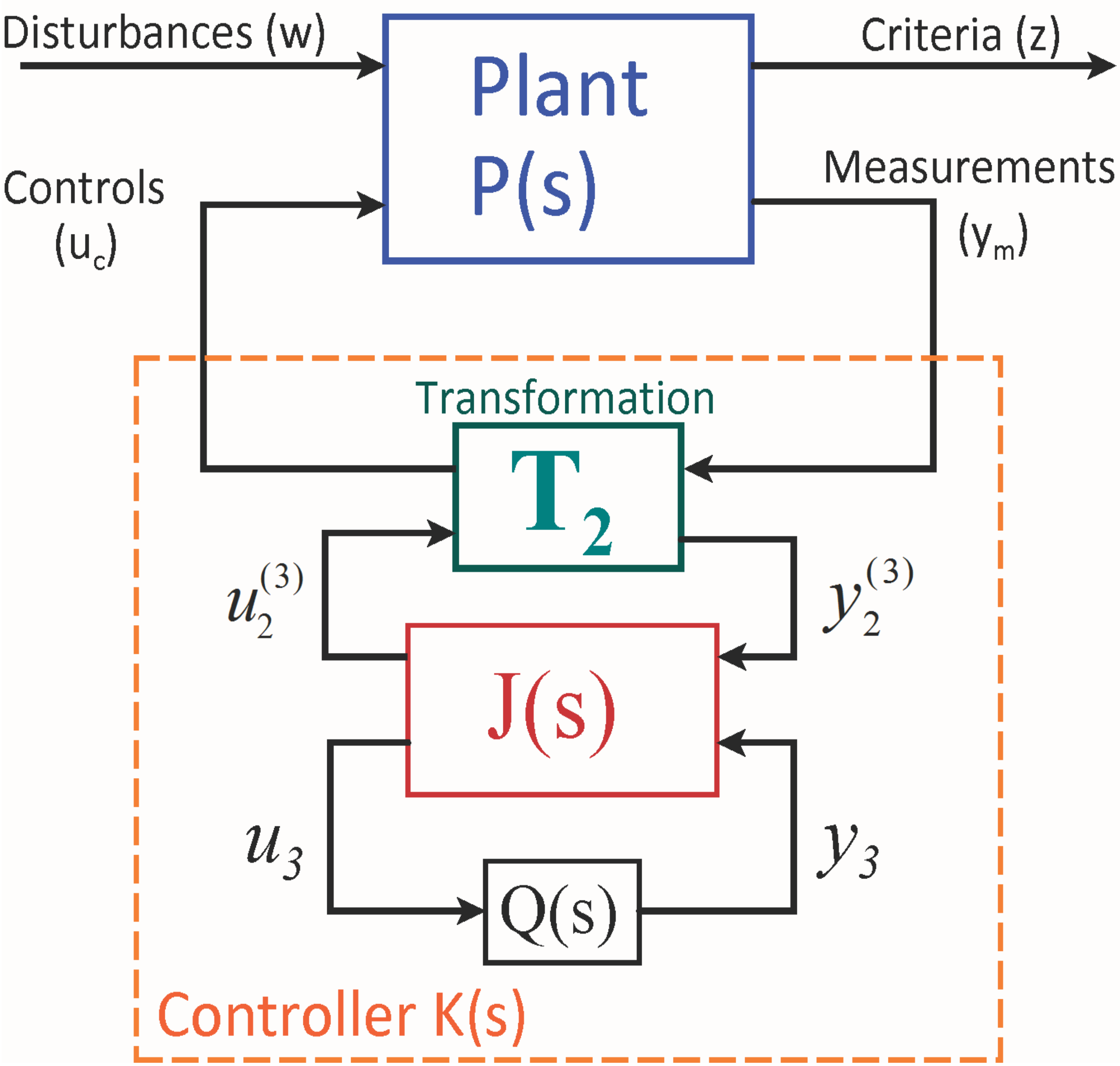


Figure 3.6 The Controller J(s) is Back-Transformed to K(s) in order to Match the Original Plant

4.0 Setting up the H_∞ Synthesis Model

In this section we describe how to set up an H_∞ synthesis model from a given plant in state-space form. The complexity of the SM is determined by the selected plant and the design requirements. In its simples form the Synthesis Model is shown in Figure 4.1. It consists of a multi-variable state-space representation of the plant that has already been prepared and it includes the necessary inputs and outputs. The inputs should include controls, disturbances, and uncertainty inputs. The outputs should include measurements, performance criteria, and uncertainty outputs. The uncertainty inputs and outputs represent internal parameter variations as we shall see in Section 5. The plant system may also include loop-shaping filters which are used to enhance performance in some of the system variables by the H_∞ optimization. They are temporarily included in the plant model for the purpose of developing the SM. They are eventually moved in the controller side where they belong when the control design is complete. However, the filters increase the plant and SM states and also the controller complexity.

The SM is created the from the plant model by picking and categorizing some inputs and some outputs. Some of the inputs will be chosen to be controls and some will be used as disturbances. Some of the outputs will become sensor measurements and some outputs will represent criteria to be minimized. Some of the columns of plant matrices B and D are selected to create the SM matrices B_1, D_{11} and D_{21W} that describe the input disturbances, and some of the columns of B and D are selected to form matrices B_2, D_{12U} and D_{22} that describe the control inputs. Similarly, some of the rows of plant matrices C and D are selected to form SM matrices C_1, D_{11} and D_{12U} that define the optimization criteria and some of the rows of matrices C and D are selected to form matrices C_2, D_{21W} and D_{22} that describe the measurements.

There is some symmetry in the SM in Figure 4.1. The size of measurement noise input w_o is equal to the number of output measurements y_m. The measurement noise is used in the estimator design. It defines the reliability of the measurement. The input criteria z_i is used to penalize the controls and its size is equal to the number of controls u_c. The two square matrices D_{12} and D_{21} are not necessarily considered to be plant dynamics but they are adjusted by the designer to optimize the control system performance and they are usually diagonal matrices. Matrix D_{12} penalizes the control input u_c and it limits the controller bandwidth. Similarly, matrix D_{21} introduces uncertainty in the measurements and prevents high gains and bandwidth in the estimator.

The input/ output vector pairs (w_i) and (z_o) do not only consist of external disturbances and criteria variables. Regulated outputs can also be included in the disturbances and criteria variables as we shall see in Section 4.1. Plant uncertainties can also be included as vector pairs in the w_i and z_o vectors. In Section 5 we describe the IFL method where each plant uncertainty can be represented with one additional input/ output pair included in the SM. This allows us to synthesize controllers which are robust to real (structured) parameter uncertainties. Figure 5.2 shows how the disturbance and criteria vectors are augmented with the inclusion of the uncertainty inputs and outputs. In section 4.2 we will modify the SM to include scaling gains or *"Design Knobs"* that can be adjusted between design iterations in order to achieve the desired control system performance and bandwidth. The gains scale the corresponding rows and columns of the SM matrices and adjust the relative effectiveness and criteria of various inputs or outputs in the H_∞ optimization process.

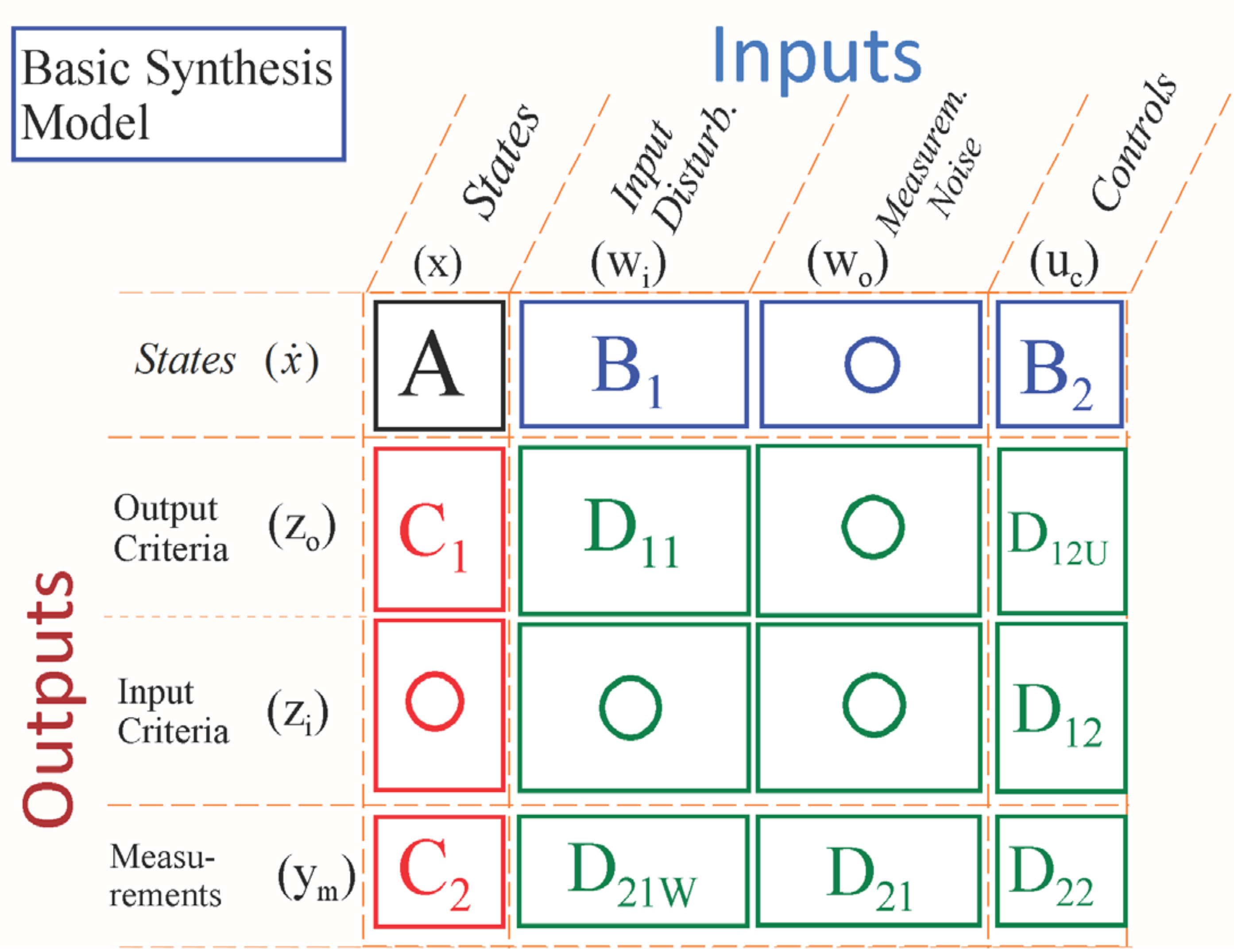


Figure 4.1 Basic Synthesis Model

4.1 Including Some Outputs to be Regulated with Commands

In Figure 4.1, the SM has no commands inputs and the vector w_i is an input disturbance. In the situation where we have some outputs z_R which are directly commanded by tracking command inputs w_c , in this case we want to achieve a small error and minimize the error $z_{re}=\{z_R-w_c\}$ by including it in the criteria vector. The command w_c is also included in the disturbances, as shown in Figure 4.2. The matrices C_1, D_{11} and D_{12U} now consist of two sets of rows that define criteria to be minimized. The upper part creates a set of criteria z_o similar to the criteria of Figure 4.1 and a lower part consists of the regulated variable errors z_{re}. In comparison with the basic SM of Figure 4.1 the disturbance vector w is now augmented to include the input commands w_c in addition to the disturbances w_i and noise the w_o. Similarly, the criterion output vector is also augmented and it consists of three parts: z_o as before, the regulated output errors $z_{or}=\{z_R-w_c\}$, and the control input criterion z_i as before. The SM augmented with regulated outputs is shown in Figure 4.2.

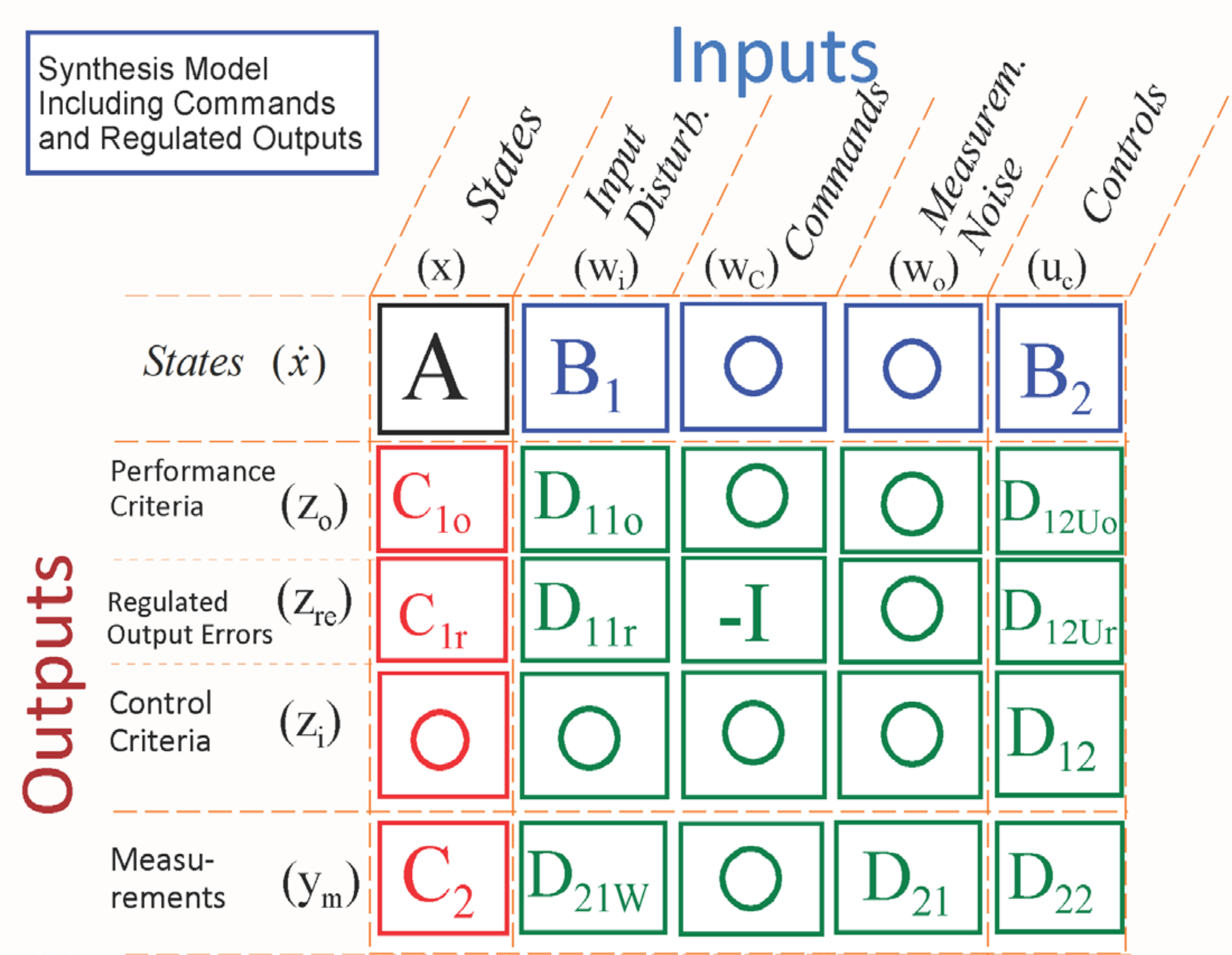


Figure 4.2 Synthesis Model Augmented with Input Commands and Regulated Outputs

4.2 Normalizing the Synthesis Model with Scaling Gains

The H-infinity control design is often an iterative process. We begin with a set of design parameters in the SM, calculate the controller, analyze the control system stability and performance, and if the control amplitudes are too big or if the performance of some variables is poor in response to commands or to disturbances, we adjust the design parameters accordingly and repeat the process until the robustness versus performance criteria are satisfactory. For example, if the controller bandwidth and gain in one of the control loops is high, we penalize the corresponding element in the z_i vector more severely to reduce the gain next time. If the estimator gain in one of the estimation loops is high, we should increase the amount of noise introduced in the SM measurements w_o. If we want to improve the performance in some of the output variables in the criteria vector z_o we must increase the corresponding gain at that output.

This adjustment is accomplished by introducing gains that scale the SM accordingly. There are 6 sets of gain vectors, 3 sets that multiply the inputs to the SM, and 3 sets that divide the outputs, as shown in figure (4.3). Those gains are not part of the original plant model but they are only used as design knobs for adjusting the relative importance (weight) of certain elements within a vector versus others in the H_∞ algorithm. They are inserted in six places in the SM. At the three inputs: disturbance (w_i), commanded outputs (w_c), and measurement noise (w_o). Also, at the three outputs: performance criteria (z_o), output regulation error (z_{re}), and control criteria (z_i). They are modified during the design process as necessary to optimize the control system performance. The gains are then absorbed in the SM for the next cycle by scaling the corresponding rows and columns of the SM matrices. Each gain vector is described in detail below.

1. The input disturbance gain $\mathbf{G_{wi}}$ is used to multiply the input disturbances (w_i). Increasing the magnitude in some of its elements, it will improve in general the system's sensitivity to those excitations at the expense of performance deterioration in other variables. The gain $\mathbf{G_{wi}}$ is initially set to the maximum expected magnitudes of the corresponding input disturbances.
2. The command scaling gain $\mathbf{G_{wc}}$ is used to multiply the command (w_c) of a regulated output, such as vehicle attitude. Increasing its magnitude will improve the command following performance of the corresponding regulated output at the expense of performance in other variables. It is initially set to the magnitude of the maximum expected command.
3. The measurement noise gain $\mathbf{G_{wo}}$ is used to define the amount of disturbance $\underline{w}_o$ corrupting the corresponding measurements. Small magnitudes in $\underline{w}_o$ will in general produce high estimator bandwidth and gains. It also expresses the amount of relative reliability in the corresponding measurement element in vector ($\underline{y}_m$), in comparison with other elements. If one element of y_m is less reliable than other elements, a heavier gain factor should be placed in that element position in vector $\underline{G_{wo}}$. The gain elements multiply the input $\underline{w}_o$ and they are initially set to the largest noise magnitude expected at the corresponding measurement.
4. The output criterion gain $\underline{\mathbf{G_{zo}}}$ divides the performance criterion output vector ($\underline{z}_o$) and adjusts the relative performance of each output relative to others. Good performance means small responses to excitation inputs. To improve, for example, the performance of a certain output in the criterion vector ($\underline{z}_o$) the inverse of the corresponding element $1/G_{zo}$ must be large. The elements of the gain vector $\underline{G}_{zo}$ are initially set to the largest permitted magnitudes of the corresponding output criteria $\underline{z}_o$. If some amplitudes are exceeded in simulations due to excessive excitations, the corresponding $\underline{G}_{zo}$ gains must be reduced, which produces heavier penalization in the optimization.
5. A similar logic applies for the regulated output errors vector. The outputs ($\underline{z}_{re}$) are divided by the scaling gains $\underline{\mathbf{G}}_{\mathbf{zr}}$ which adjust the amount of errors in the regulated outputs $\underline{z}_{re}$. The error in a regulated output is reduced by increasing the gain $1/\underline{G}_{zr}$ in the corresponding output. The elements of the gain vector $\underline{\mathbf{G}}_{\mathbf{zr}}$ are initially set to the magnitudes of the largest allowable output errors in $\underline{z}_{re}$. The smaller the expected error the smaller the $\underline{G}_{zr}$.
6. The controls performance criterion (z_i) penalizes the controls and the gain $\underline{\mathbf{G}}_{\mathbf{zi}}$ adjusts the control system bandwidth which affects also the amplitudes of the control inputs $\underline{u}_c$. The criterion (z_i) is scaled by dividing it with the gains vector $\underline{\mathbf{G}}_{\mathbf{zi}}$. Initially, the control criteria are defined by the vector $\underline{z}_i=D_{12}\,\underline{u}_c$, where the matrix D_{12} is set equal to unity and the elements of gain $\underline{G}_{zi}$ are set to the largest expected magnitudes of the controls $\underline{u}_c$. The gains are checked in simulations and if some of the controls exceed the maximum allowable amplitudes the corresponding gains in $G_{\mathbf{zi}}$ must be reduced which increases its penalization in the algorithm.

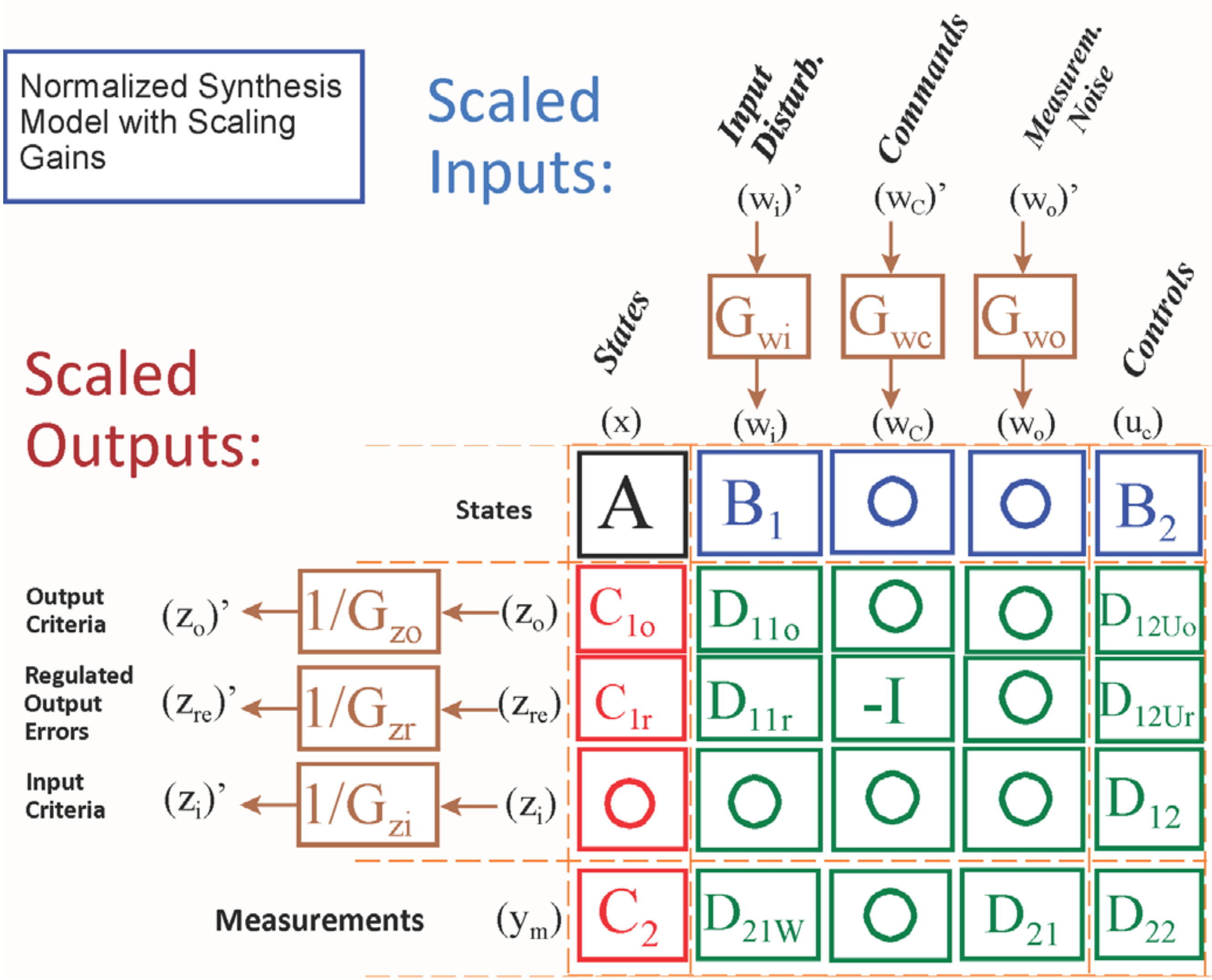


Figure 4.3 Normalized SM Including Scaling Gains of Maximum Inputs and Maximum Outputs

4.3 Including Parameter Uncertainties in the SM

In Section 5 we describe how internal parameter variations in the design model can be characterized by additional inputs and outputs which are normalized and they hypothetically connect to a unitized Δ block. The Synthesis Model of Figure 4.3 is now combined with Figure 5.2, as shown in Figure 4.4. The additional inputs (w_p) are treated as disturbances and the additional outputs (z_p) are grouped with the criteria. There are no additional states. There are no gains included to scale the uncertainty inputs and outputs because the uncertainty model is already normalized for a unity Δ block. The parameter uncertainties and the regulated outputs are optional and they not always included in the SM.

In Figure 4.4 the scaling gains G_{wi}, G_{wc}, G_{wo}, G_{zi}, G_{zre}, and G_{zo} that were described in section 4.2 attempt to normalize the sensitivity function from the combined disturbance vector (w_i w_c w_o) to the combined criteria vector ($\underline{z}_i$ $\underline{z}_{re}$ $\underline{z}_o$) to be less than one. The uncertainty plant in Figure 5.1 is already scaled by the IFL process and the uncertainty inputs w_p and outputs z_p are already normalized to connect with a unit-diagonal block, where $\|\Delta\| \leq 1$. In fact, the entire SM is normalized.

The H_∞ controller closes the loop between the measurements (y_m) and the controls (u_c) and it attempts to achieve the following: Reduce the infinity norm between the combined disturbance vector ($\underline{w}$) and the combined criterion vector ($\underline{z}$) to be less than a certain upper bound (γ), meaning, robustness to parameter uncertainties, command following, performance against external disturbances, and of course good stability margins in the presence of known parameters variations in the uncertain plant.

Robust performance is achieved when the H-infinity norm of the closed-loop sensitivity function between the combined input vector (w_p w_i w_c w_o) and the combined output vector ($\underline{z}_p$ $\underline{z}_i$ $\underline{z}_{re}$ $\underline{z}_o$) is less than one at all frequencies. It means that the control system derived from this formulation, not only minimizes the sensitivity transfer function between the disturbances and criteria (both at the plant input and output for good performance) but also takes into consideration variations in the uncertain plant parameters and attempts to maintain both stability and performance despite variations.

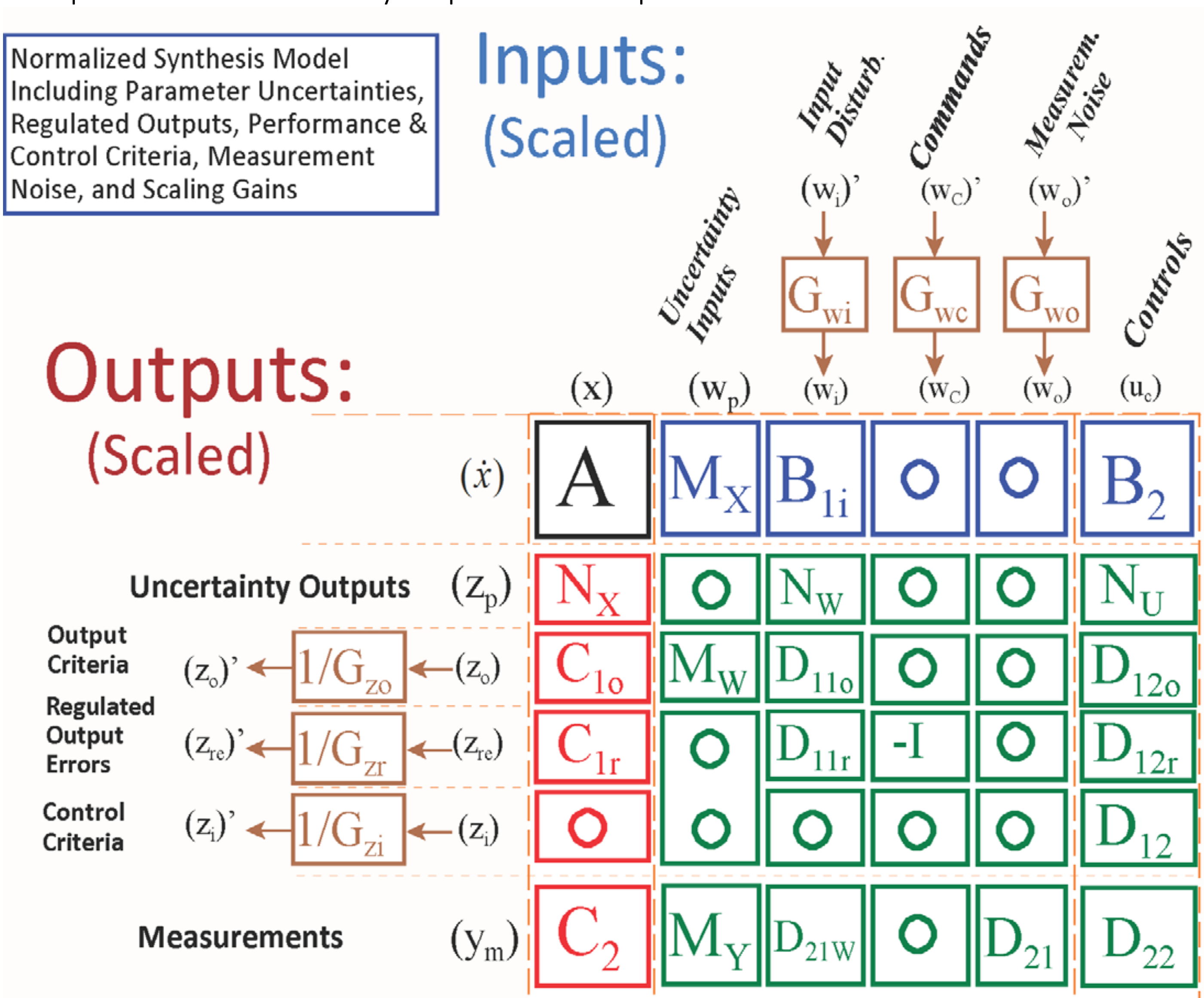


Figure 4.4 Synthesis Model Augmented with Parameter Variation Inputs and Outputs Derived from the IFL Method

5.0 The Internal Feedback Loop (IFL) Structure

The IFL method allows internal parameter perturbations in a system to be treated like external disturbances by introducing fictitious inputs and outputs. This representation allows us to use μ-tools for analyzing robustness to uncertainties or to apply H_∞ and other robust methods to design control systems that can tolerate a certain amount of parameter variations. To utilize the IFL concept the system must be expressed in the following form, where [ΔA, ΔB, ΔC, ΔD] are variations in the state-space system matrices as a result of variation in one of the parameters.

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} + \begin{bmatrix} \Delta A & \Delta B \\ \Delta C & \Delta D \end{bmatrix} \right\} \begin{bmatrix} x \\ u \end{bmatrix}$$

Suppose that they are (l) independently perturbed parameters: p_1, p_2, ... p_l, with bounded parameter variations δp_i, where their magnitude $|\delta p_i| \le 1$. The perturbation matrix ΔP= [ΔA, ΔB; ΔC, ΔD] can be decomposed with respect to each parameter variation as follows:

$$\Delta_i = -\sum_{i=1}^{l} \delta p_i \begin{pmatrix} \alpha_x^{(i)} \\ \alpha_y^{(i)} \end{pmatrix} \begin{pmatrix} \beta_x^{(i)} & \beta_u^{(i)} \end{pmatrix}$$

Where for each parameter p_i

$\alpha_x^{(i)}$ and $\alpha_y^{(i)}$ are column vectors

$\beta_x^{(i)}$, and $\beta_u^{(i)}$ are row vectors

The plant uncertainty matrix ΔP due to all perturbations can be written in the following form, where the perturbation block ΔP is assumed to have a rank-1 dependency with respect to each parameter p_i.

$$\Delta P = -\begin{pmatrix} M_x \\ M_y \end{pmatrix} \Delta \begin{pmatrix} N_x & N_u \end{pmatrix} = -M \Delta N$$

Where M_x and M_y are stacks of column vectors, and N_x and N_u are stacks of row vectors as shown below

$$M_x = \begin{bmatrix} \alpha_x^{(1)} & \alpha_x^{(2)} & \dots & \alpha_x^{(l)} \end{bmatrix}; \qquad M_y = \begin{bmatrix} \alpha_y^{(1)} & \alpha_y^{(2)} & \dots & \alpha_y^{(l)} \end{bmatrix}$$

and

$$N_x = \begin{bmatrix} \beta_x^{(1)} \\ \vdots \\ \beta_x^{(l)} \end{bmatrix}; \qquad N_u = \begin{bmatrix} \beta_u^{(1)} \\ \vdots \\ \beta_u^{(l)} \end{bmatrix}$$

Where Δ = diag [δp_1, δp_2, δp_3,.... δp_l] is the diagonal block of Figure 5.1 containing the uncertainties. Notice, that in order to simplify the implementation, the columns of matrices M_x and M_y and the rows of matrices N_x and N_u are scaled, so that the elements of the diagonal block Δ have unity upper bound. Now let us introduce two new variables (z_p and w_p) and rewrite the equations in the following system form in order to express it as a block diagram.

$$z_p = N_x x + N_u u \quad and \quad w_p = -\Delta z_p$$

The perturbed state-space system can be written in the following augmented representation which is the same as the original system in the upper left side, with some additional input and output vectors, an input and an output for each parameter uncertainty.

$$\begin{pmatrix} \dot{x} \\ y \\ z_p \end{pmatrix} = \begin{bmatrix} A & B & M_x \\ C & D & M_y \\ N_x & N_u & 0 \end{bmatrix} \begin{pmatrix} x \\ u \\ w_p \end{pmatrix}$$

If we further separate the plant inputs (u) into disturbances (w) and controls (u_c), that is, u=[w, u_c], and if we also separate the plant outputs (y) into performance criteria (z) and control measurements (y_m), the above system is augmented as shown below.

$$\begin{pmatrix} \dot{x} \\ z \\ y_m \\ z_p \end{pmatrix} = \begin{bmatrix} A & B_1 & B_2 & M_x \\ C_1 & D_{11} & D_{12} & M_w \\ C_2 & D_{21} & D_{22} & M_{ym} \\ N_x & N_w & N_{uc} & 0 \end{bmatrix} \begin{pmatrix} x \\ w \\ u_c \\ w_p \end{pmatrix}$$

The above formulation is used for μ-synthesis or robustness analysis using μ-methods. It is also shown in block diagram form in Figure 5.1. The uncertainties block Δ is connected to the plant by means of the inputs w_p and the outputs z_p. The columns in the M_x, M_w, and M_{ym} matrices and the rows in the N_x , N_w, and N_{uc} matrices are scaled by dividing with the square root of the corresponding singular value which normalizes the elements of the uncertainty block Δ to unity. Figure 5.2 shows how the H-infinity synthesis model is augmented by the inclusion of the parameter uncertainty inputs and outputs.

The control system K(s) is designed to stabilize the plant P(s). When the feedback loop is closed between y_m and u_c the control system is also expected to keep the plant stable despite all possible variations in the elements of the block Δ which are allowed to vary between -1 and +1. This property is defined as "Robust Stability". In addition to "Robust Stability" the control system must also satisfy "Nominal Performance" which is a bounded and well-behaved response between the disturbances w and the criteria z. We also have third property for the perturbed plant which is called "Robust Performance". The plant P(s) has the control loop closed and also the uncertainty loop closed via the Δ block. The closed-loop system satisfies the "Robust Performance" property when it remains stable and it is also able to satisfy the above two performance criteria, which is, the transfer function between w and z satisfies performance despite all possible variations in the internal parameters represented in the normalized uncertainties block Δ, where the individual magnitudes δ_i do not exceed 1. This happens when the structured singular value frequency response (μ) between the combined vectors: [w, w_p] and [z, z_p] is less than 1 at all frequencies.

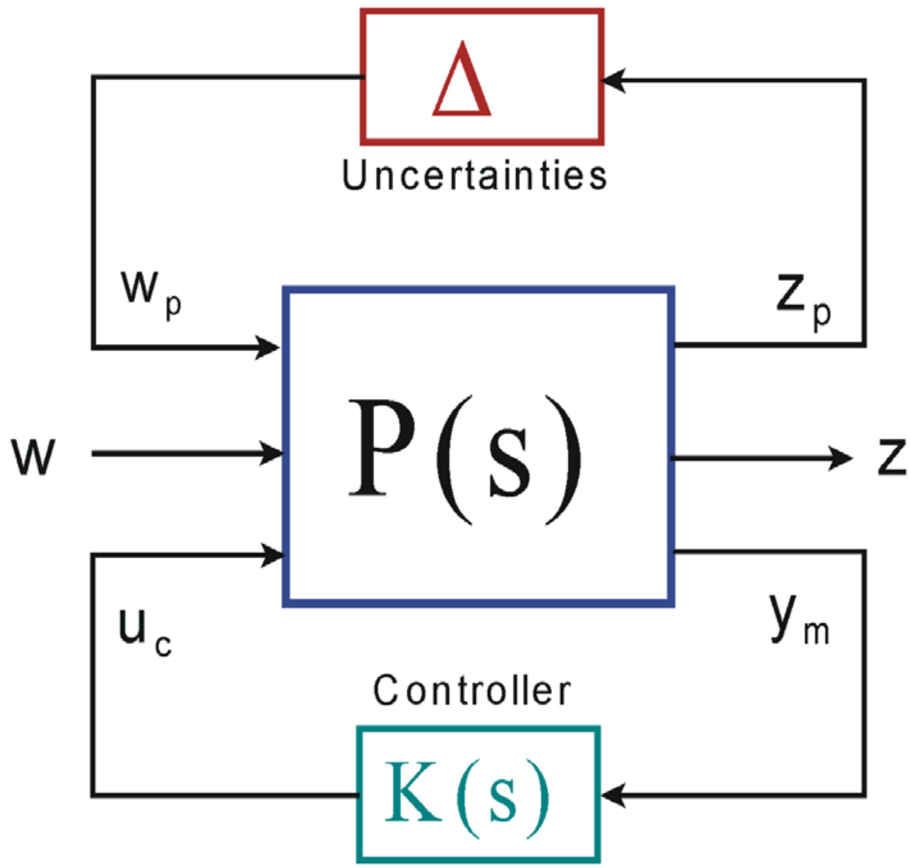


Figure 5.1 Robustness Analysis Block showing the Uncertainties IFL loop, the control feedback loop, the disturbances (w), and performance outputs (z)

This system can also be represented in matrix transfer function form as follows

$$\begin{pmatrix} z_p \\ z \\ y_m \end{pmatrix} = \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix} \begin{pmatrix} w_p \\ w \\ u_c \end{pmatrix}$$ where:

$$w_p = -\Delta z_p \qquad and \qquad u_c = -K(s)\, y_m$$

After closing the loop with a stabilizing controller K(s) the closed loop system is represented with the following transfer function matrix

$$\begin{pmatrix} z_p \\ z \end{pmatrix} = \begin{bmatrix} T_{11}(s) & T_{12}(s) \\ T_{21}(s) & T_{22}(s) \end{bmatrix} \begin{pmatrix} w_p \\ w \end{pmatrix} \quad \text{and} \quad w_p = -\Delta z_p$$

where

$$T_{11}(s) = G_{11} - G_{13}K\left(I + G_{33}K\right)^{-1}G_{31}; \qquad T_{12}(s) = G_{12} - G_{13}K\left(I + G_{33}K\right)^{-1}G_{32}$$

$$T_{21}(s) = G_{21} - G_{23}K\left(I + G_{33}K\right)^{-1}G_{31}; \qquad T_{22}(s) = G_{22} - G_{23}K\left(I + G_{33}K\right)^{-1}G_{32}$$

The above transfer functions are used to analyze robustness and performance of the closed loop system

Robust Stability: Stability robustness with respect to parameter uncertainty is determined by the transfer function $T_{11}(s)$. Smaller $\|T_{11}\|_\infty$ allows larger parameter uncertainty for closed loop stability. The closed loop system is considered to be robustly stable with respect to the parameter perturbations block Δ, where $\|\Delta\| \leq 1$, when the $\mu\{T_{11}(\omega)\} < 1$ at all frequencies (ω).

Nominal Performance: Nominal performance is used to calculate the system's sensitivity to excitations and it is obtained from the transfer function $T_{22}(s)$. This transfer function must be scaled by multiplying its inputs with the max magnitude of the excitations and by dividing its outputs with the max allowable error. The system satisfies Nominal Performance when the scaled $\|T_{22}(\omega)\|_\infty < 1$ at all frequencies (ω). For example, maximum wind-gust velocity disturbance must not exceed the maximum allowable dispersion in angle of attack.

Robust Performance: is achieved when the system meets the performance and robustness requirements together. This happens when the following condition is satisfied at all frequencies.

$$\mu \begin{bmatrix} T_{11}(s) & T_{12}(s) \\ T_{21}(s) & T_{22}(s) \end{bmatrix} < 1$$

5.1 Parameter Uncertainties Modeling Program

There is a Flixan program that implements the IFL method and it can be used to create the additional inputs and outputs in a flight vehicle system that model the internal parameter variations. The fictitious inputs and outputs theoretically connect with the normalized uncertainty block Δ, as shown in Figure 5.1, and it is assumed that each element of the diagonal uncertainty block Δ can vary between ±1. The IFL program calls the flight vehicle modeling program that processes the vehicle data from an input file and generates state-space systems. In addition to the vehicle dataset, the program also reads the uncertainties from a separate dataset which is also included the same input file (.Inp). The algorithm calls the vehicle modeling program multiple times and processes the uncertainties together with the vehicle data. It begins by processing the nominal vehicle dataset and repeats the data processing for each parameter variation. It eventually generates the uncertainty state-space system which is similar to the nominal system but it includes the additional input/ output pairs which are supposed to connect with the extracted uncertainty block Δ. The following algorithm describes the process of calculating the uncertainty system:

1. The modeling program is used initially to process the nominal set of vehicle data and to create the "known" plant state-space model [A, B; C, D].
2. One (and only one) of the vehicle data parameters must be modified at a time, either increased or decreased from its nominal value by an amount that is equal to the maximum expected variation (δp_1) and the vehicle data is reprocessed by the vehicle modeling program to create a new state-space system [A_1, B_1, C_1, D_1] that corresponds to parameter #1 variation. The matrix difference between the nominal and the perturbed state-space models is calculated:
$$\begin{bmatrix} \Delta A_1 & \Delta B_1 \\ \Delta C_1 & \Delta D_1 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} - \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$
3. This matrix is decomposed using SVD to calculate the column vectors $\alpha_x^{(1)}$ and $\alpha_y^{(1)}$ and the row vectors $\beta_x^{(1)}$, and $\beta_u^{(1)}$, as shown in the equation.
4. Restore the previous parameter to its original value and modify another parameter #2 in the vehicle input data by an amount δp_2 that represents the maximum variation of this parameter, as in step-2. Repeat steps 2 and 3 and calculate the vectors $\alpha_x^{(2)}$, $\alpha_y^{(2)}$, $\beta_x^{(2)}$, and $\beta_u^{(2)}$.
5. Select another parameter to perturb and repeat steps 2, and 3 until there are no more uncertain parameters to vary. Stack the row and column vectors as shown to create the stacks of column vectors: M_x and M_y and the stacks of row vectors: N_x and N_u.
6. These matrices are then used to create the additional inputs and outputs in the state-space model. The columns of matrices M_x and M_y and the rows of matrices N_x and N_u must also be scaled according to the magnitude of the uncertainties δp_i so that the interconnections correspond to a unity normalized Δ-block.

The uncertainty model is then used in combination with the flight control system to analyze the closed-loop system performance and robustness to uncertainties by calculating the μ-frequency response of the plant across the interconnections with the Δ block, as shown in Figure 5.1. That is, between w_p and z_p, with the control loop K(s) closed.

The parameter uncertainties data-set in the input file is similar to the vehicle dataset. It includes variations from the nominal vehicle data and a title above the data. The variations should correspond to the parameters in the vehicle dataset. There should be the same number of aerosurfaces, engines, slosh tanks, etc. Only the variations in the uncertain parameters should be non-zero. Obviously, the variations in the parameters which are known and do not vary must be set to zero. An additional input/ output pair is created in the system for each uncertainty. In some cases, two connections are created for one parameter variation, such as the X_{CG}, which affects both pitch and lateral axes. In this case the pitch and lateral systems must be decoupled, and one I/O pair is associated with the pitch system and the other I/O pair is associated with the lateral system.

The Flixan program identifies datasets that contain parameter uncertainties from this label: "*UNCERTAIN PARAMETER VARIATIONS FROM NOMINAL* …" which is located above the dataset. There is also a title below this label which identifies a particular uncertainties dataset, similar to all other types of Flixan datasets. This title associates the uncertainties data with a vehicle input data, and it must be included at the bottom of the vehicle input data in order to associate the variations with the actual vehicle parameters.

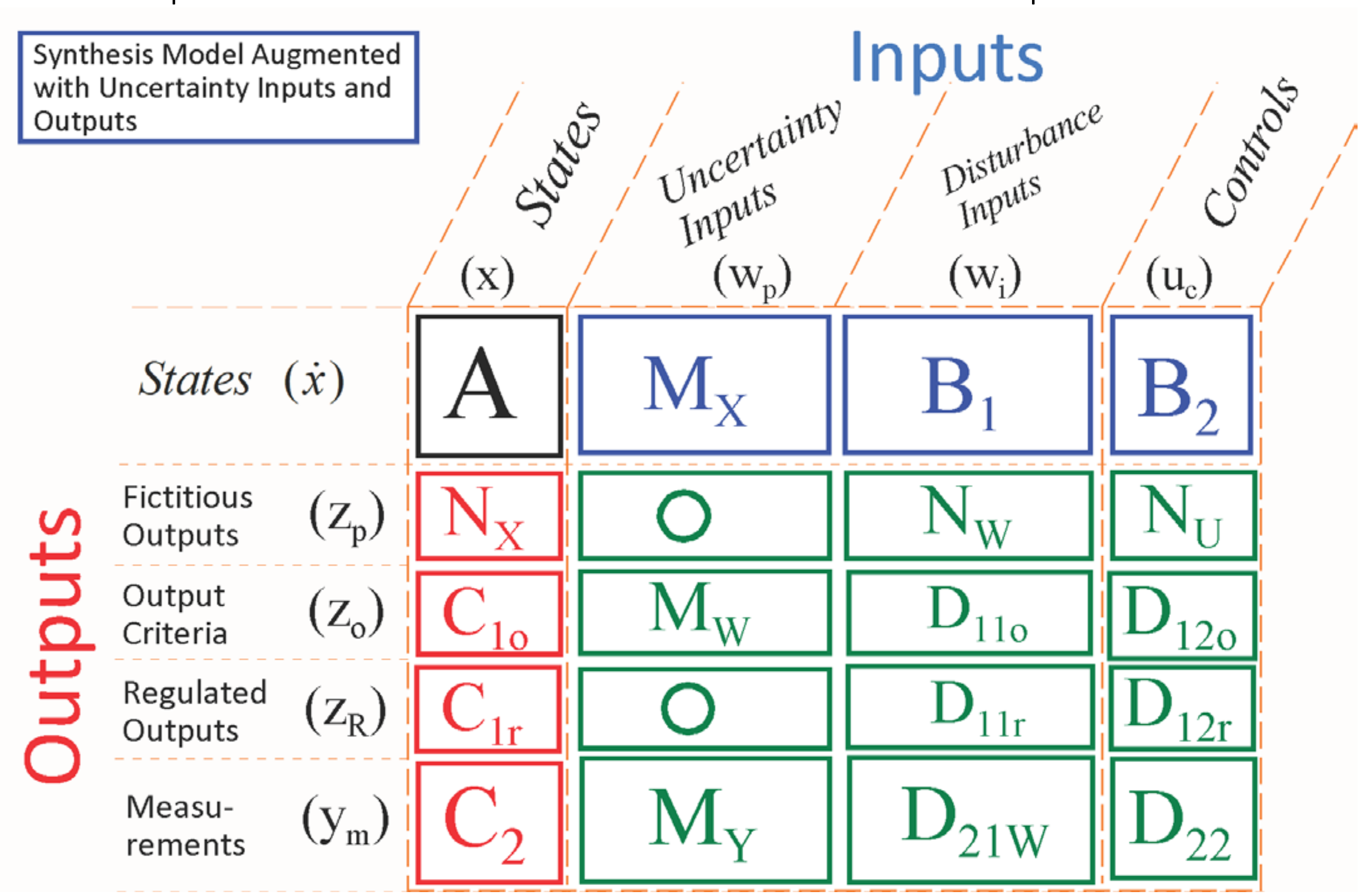
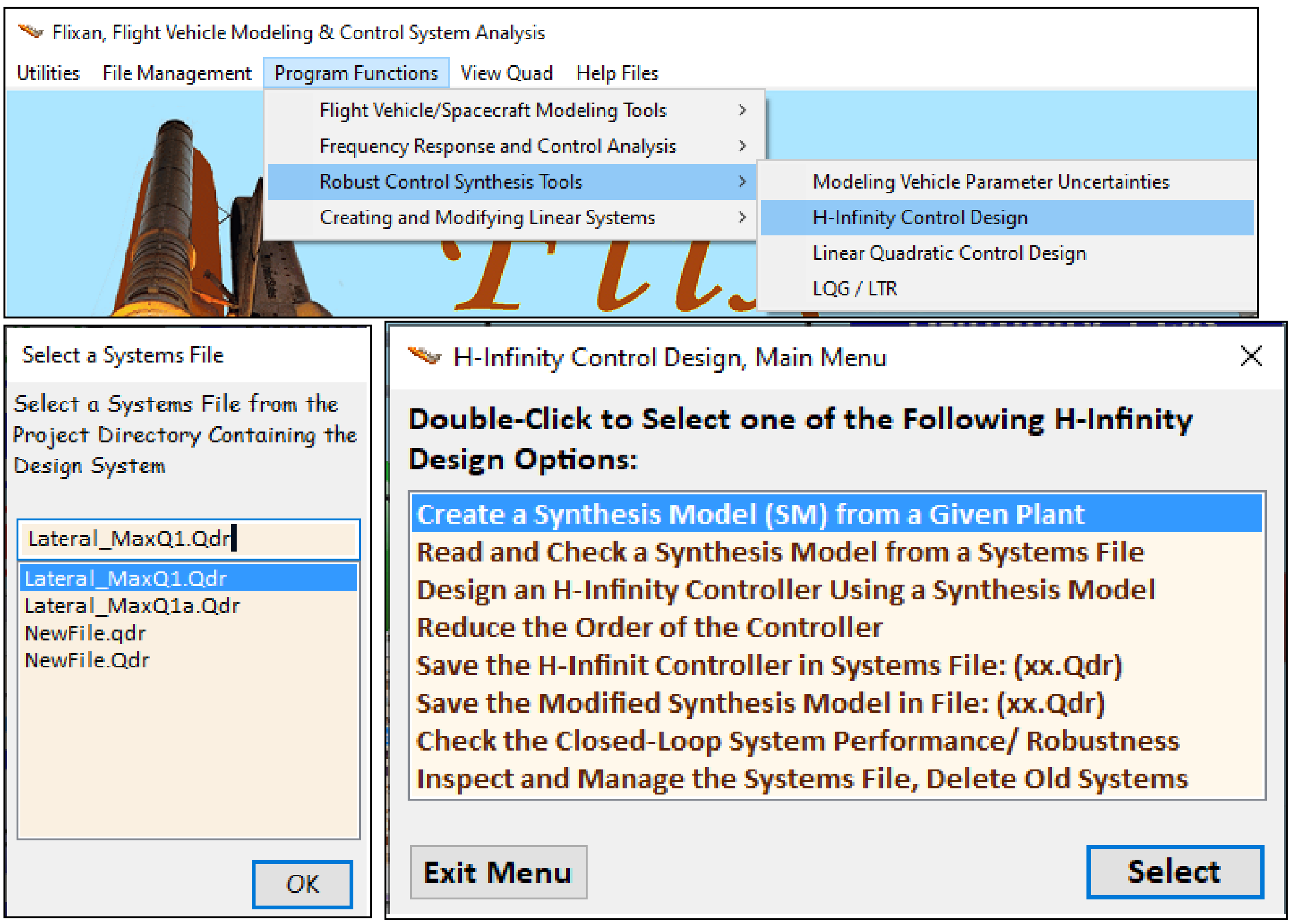


Figure 5.2 Synthesis Model Augmented with Uncertainty Inputs (w_p) and Uncertainty Outputs (z_p)

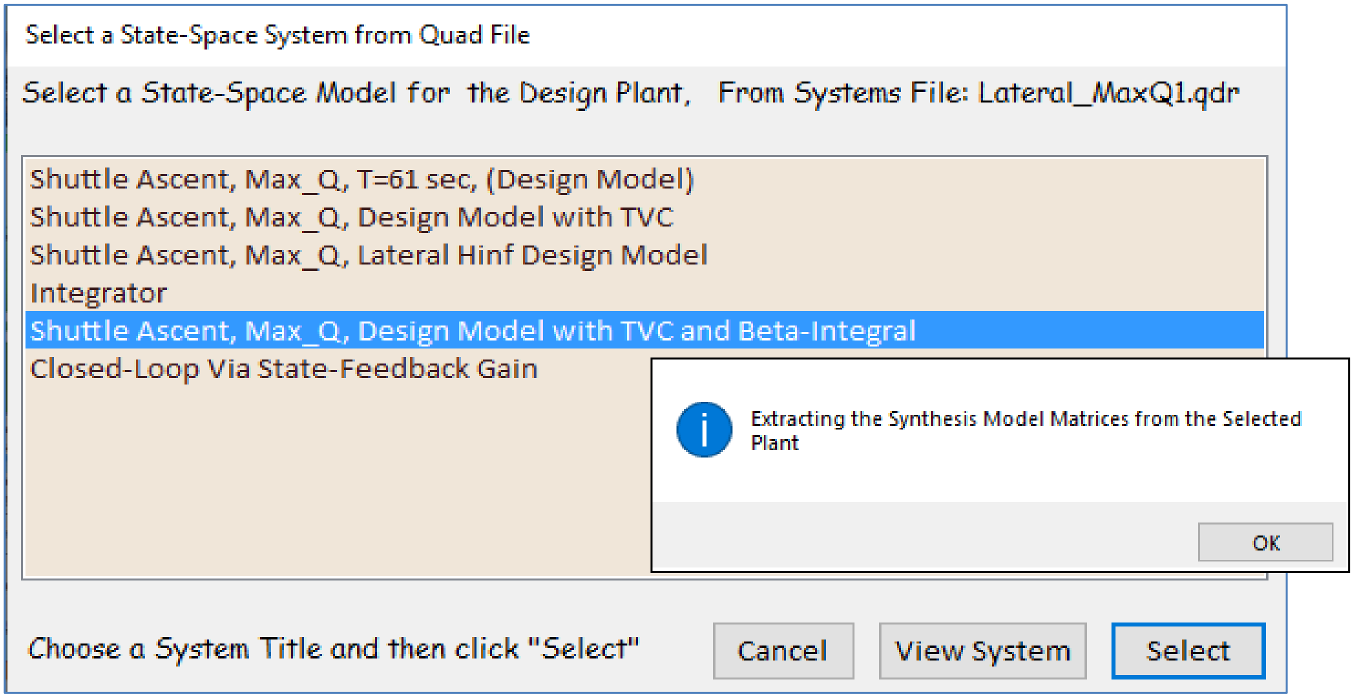
6.0 Running the H-Infinity Program

The H-infinity program includes several options. The user must first create the Synthesis Model from the plant system and then use it to design the control system. We assume that the plant system to be controlled is already created and saved in the systems file. To run the program, select "*Program Functions*" from the Flixan main menu, then "*Robust Control Synthesis Tools*", and then "*H-Infinity Control Design*", as shown below. Select also the directory that includes the system files. The next menu is for selecting the systems file (.Qdr) and from the H-infinity main menu we select the first option to create the SM.

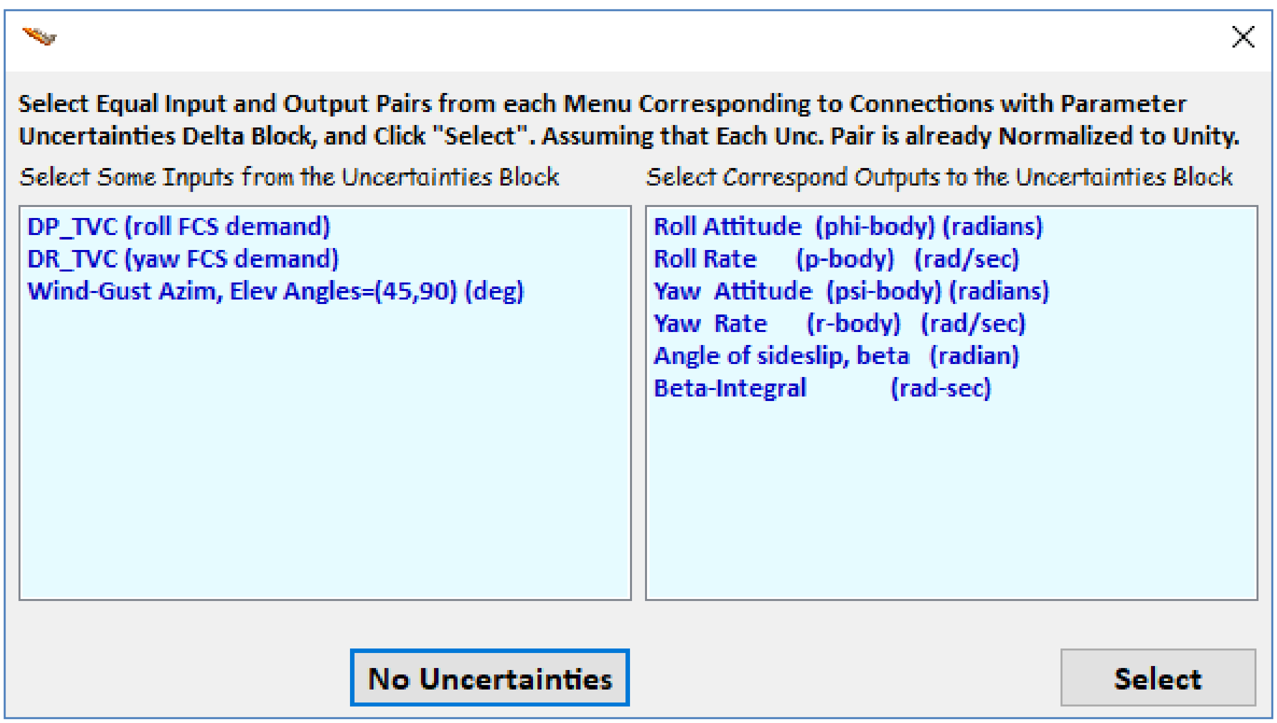


6.1 Creating the Synthesis Model

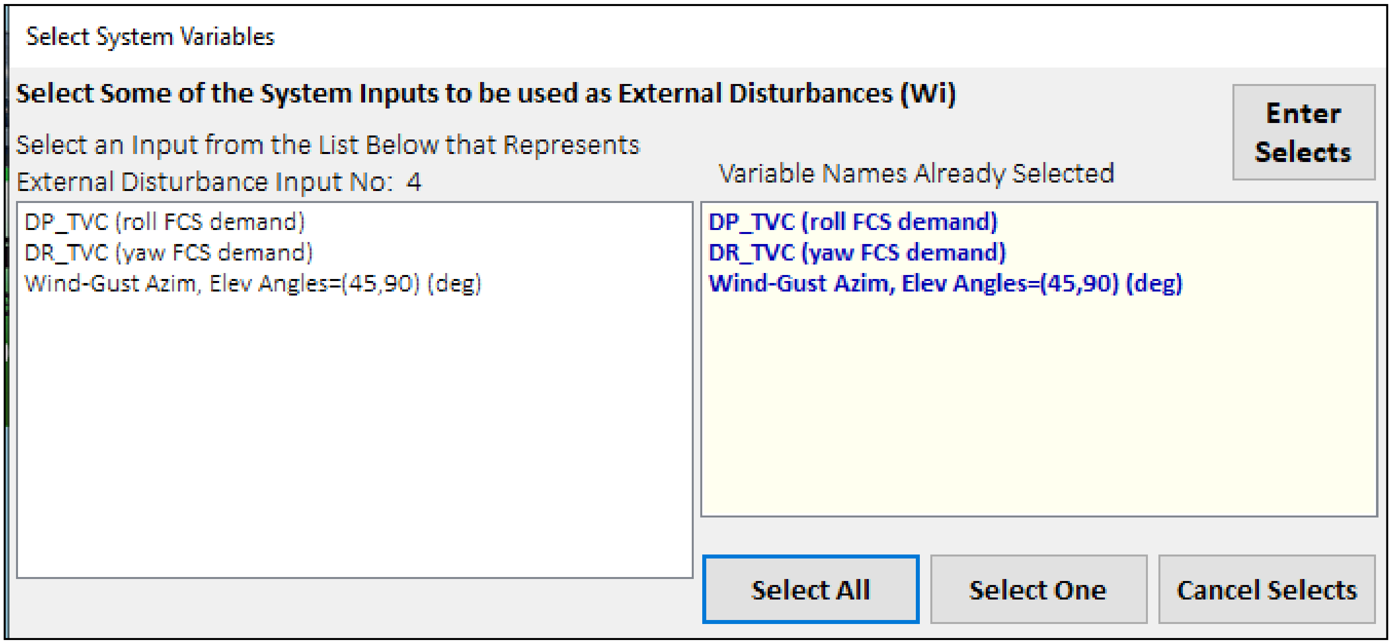
The first option is used to create the Synthesis Model from the plant system as it was described in Section 4. The SM is also a state-space system consisting of 9 matrices and it is saved in the same systems file. This interactive utility creates the SM by helping the user to define the control and disturbance inputs, the measurement and criteria outputs, and also the performance requirements. They will all be captured in the SM. From the main menu of the H-Infinity program select "*Create a Synthesis Model (SM)*" to create the SM from the plant model. The following menu shows the titles of the systems which are included in the systems file. Select the design plant and click on "*Select*". The SM will be created from this system.



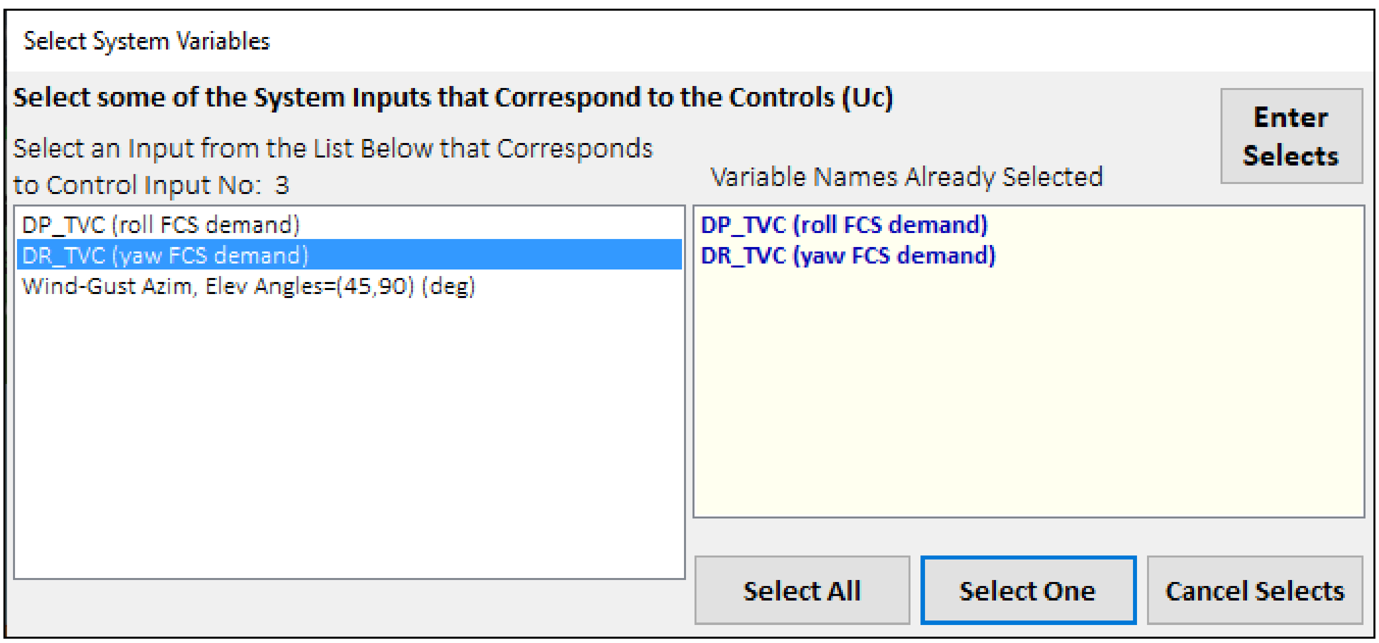
The first menu is used to define parameter uncertainties. That is, inputs and outputs that connect to the uncertainties Δ block, as described in Section 4.3. In this example we did not define any uncertain parameters and have not created any uncertainty inputs and outputs. We, therefore, click on "*No Uncertainties*" to continue.



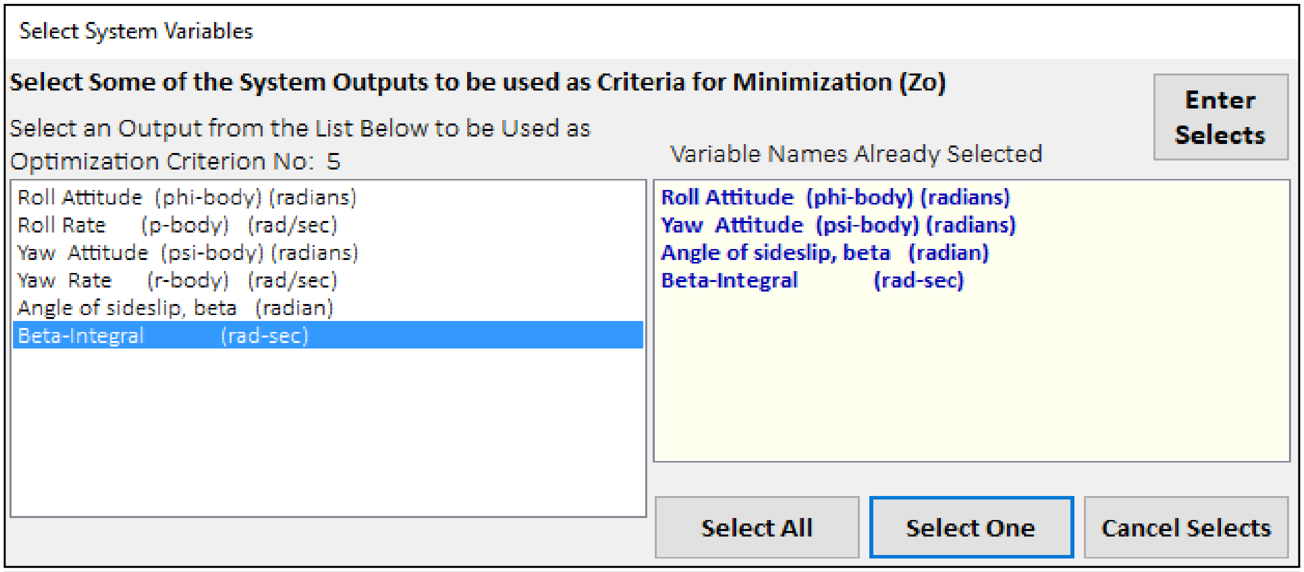
The next menu is used to define external disturbance inputs. The plant model has 3 inputs and all 3 will be considered as disturbances. Click on "*Select All*" and then on "*Enter Selects*" to continue.

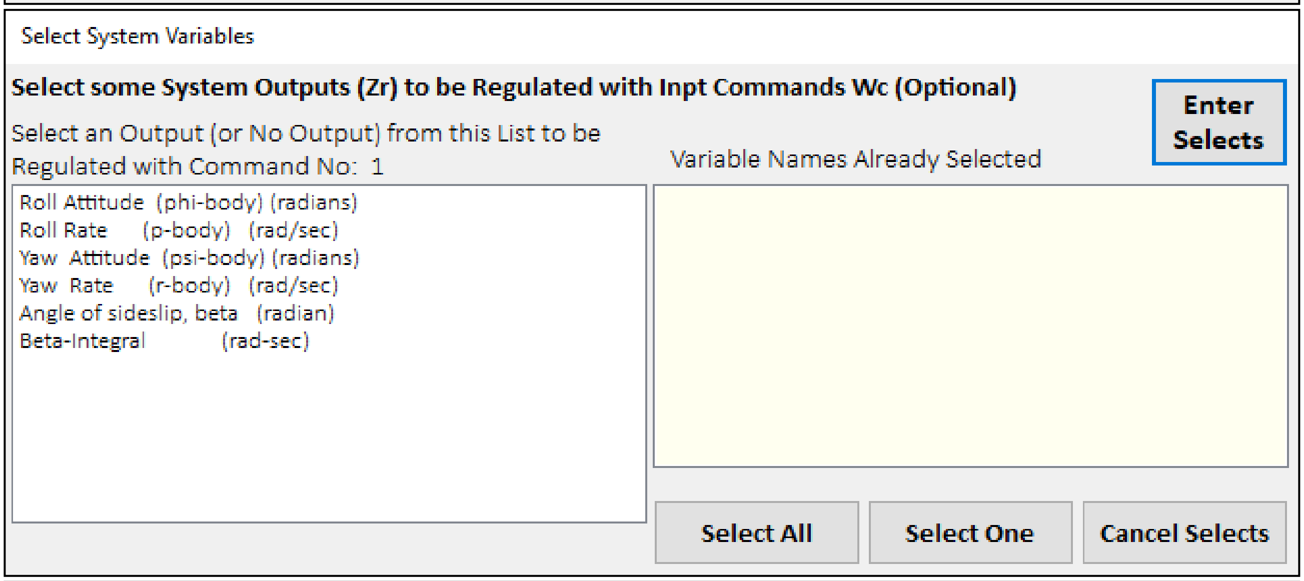


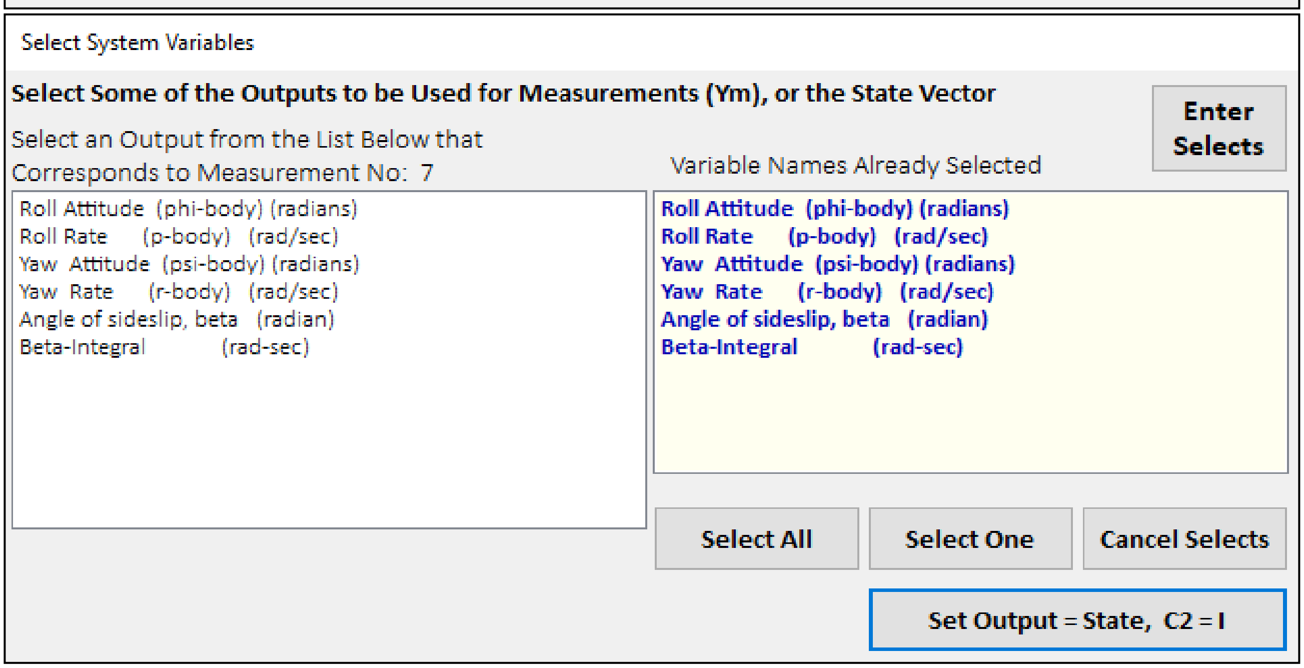
The next menu is for selecting the control inputs. There are two control inputs, roll and yaw control-demands. Select one at a time and then click on "*Enter Selects*" to continue.



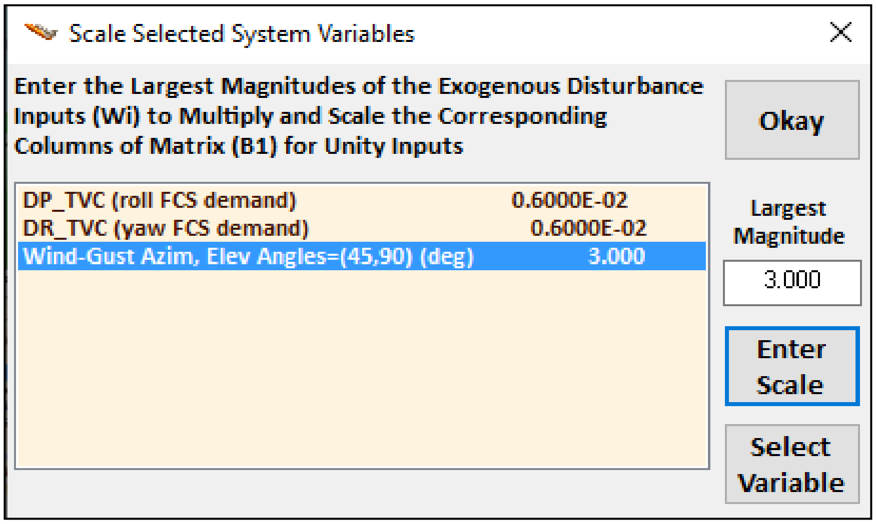
The next dialog is used for selecting outputs to be optimized. In this example the output of the plant model consists of the entire state vector of 6 variables. We will optimize only four of those state variables, the two attitudes, beta, and β-integral. Select one variable at a time and then click on "*Enter Selects*" to continue. The next menu is for selecting outputs to be regulated with input commands. In this case we do not have any. Do not select anything but click on "Enter Selects" to continue. The next menu is for selecting the output measurements. In this example the measurements are the entire state vector. Select all of them by clicking on "*Set Output= State*" and then click on "*Enter Selects*" to continue.


Select System Variables
Select Some of the System Outputs to be used as Criteria for Minimization (Zo)
Select an Output from the List Below to be Used as Optimization Criterion No: 5
Enter Selects
Variable Names Already Selected
Roll Attitude (phi-body) (radians)
Roll Rate (p-body) (rad/sec)
Yaw Attitude (psi-body) (radians)
Yaw Rate (r-body) (rad/sec)
Angle of sideslip, beta (radian)
Beta-Integral (rad-sec)
Roll Attitude (phi-body) (radians)
Yaw Attitude (psi-body) (radians)
Angle of sideslip, beta (radian)
Beta-Integral (rad-sec)
Select All
Select One
Cancel Selects

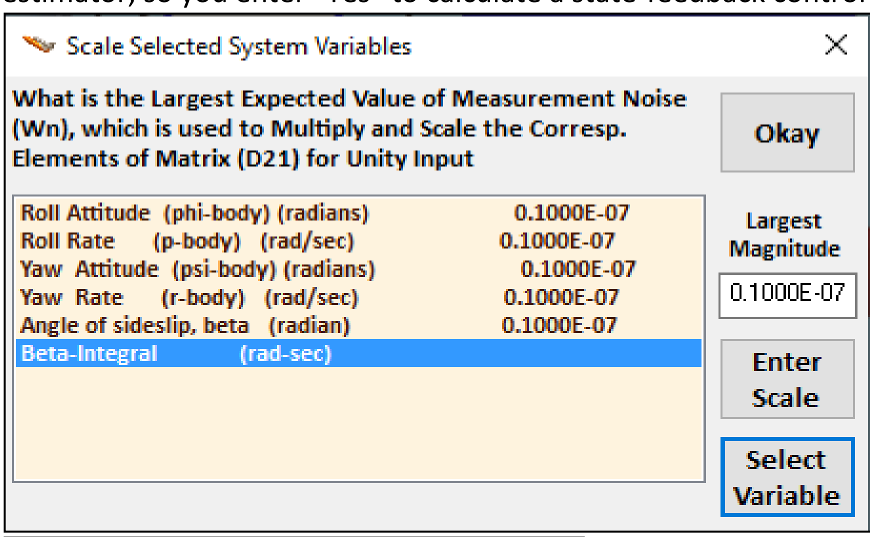

Select System Variables
Select some System Outputs (Zr) to be Regulated with Inpt Commands Wc (Optional)
Select an Output (or No Output) from this List to be Regulated with Command No: 1
Enter Selects
Variable Names Already Selected
Roll Attitude (phi-body) (radians)
Roll Rate (p-body) (rad/sec)
Yaw Attitude (psi-body) (radians)
Yaw Rate (r-body) (rad/sec)
Angle of sideslip, beta (radian)
Beta-Integral (rad-sec)
Select All
Select One
Cancel Selects

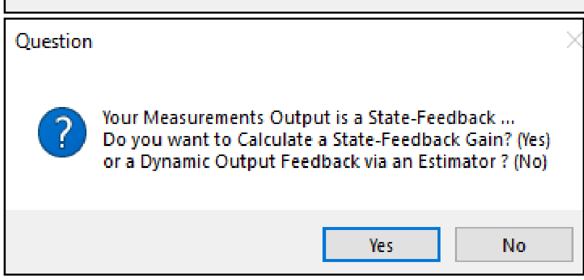

Select System Variables
Select Some of the Outputs to be Used for Measurements (Ym), or the State Vector
Select an Output from the List Below that Corresponds to Measurement No: 7
Enter Selects
Variable Names Already Selected
Roll Attitude (phi-body) (radians)
Roll Rate (p-body) (rad/sec)
Yaw Attitude (psi-body) (radians)
Yaw Rate (r-body) (rad/sec)
Angle of sideslip, beta (radian)
Beta-Integral (rad-sec)
Roll Attitude (phi-body) (radians)
Roll Rate (p-body) (rad/sec)
Yaw Attitude (psi-body) (radians)
Yaw Rate (r-body) (rad/sec)
Angle of sideslip, beta (radian)
Beta-Integral (rad-sec)
Select All
Select One
Cancel Selects
Set Output = State, C2 = I

We have now finished defining the input and output variables. The next step is to enter the gains that will be used to scale them, as described in Section 4.2. Those gains are performance parameters that can be changed in the next design iteration. The dialog below scales the disturbance inputs. Click on one input at a time to highlight it, click on "*Select Variable*", enter the scaling gain which is the maximum expected disturbance at each input, and click on "*Enter Scale*" to accept it, one at a time. The scale value appears in the menu next to the variable label. When you finish click on "*Okay*" to go to the next dialog.

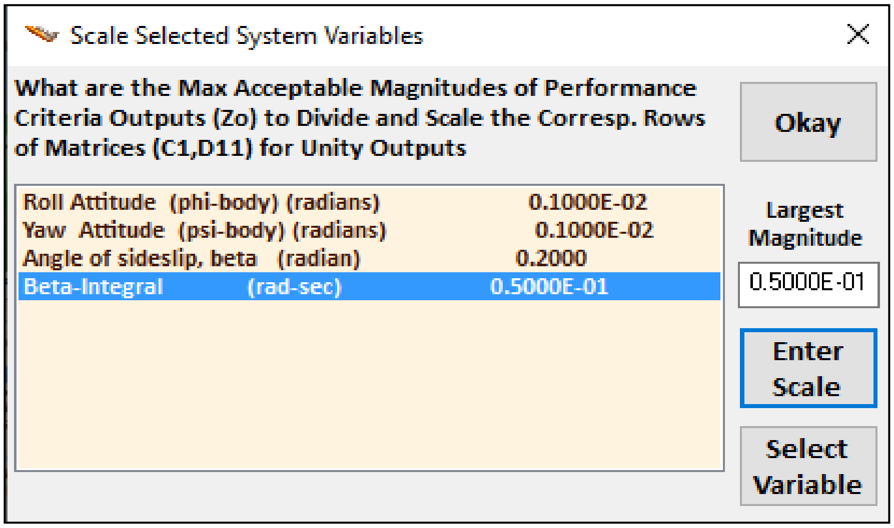


This dialog is for entering the measurement noise. In this example the measurement is the entire state-vector and we do not want to build a state estimator. We could if the measurement was noisy, but in this case we tell the program that we don't want the estimator by inserting zero or very small noise magnitude in each output/state variable. The program requires a confirmation that you do not want to create an estimator, so you enter "Yes" to calculate a state-feedback control gain and not a dynamic controller.

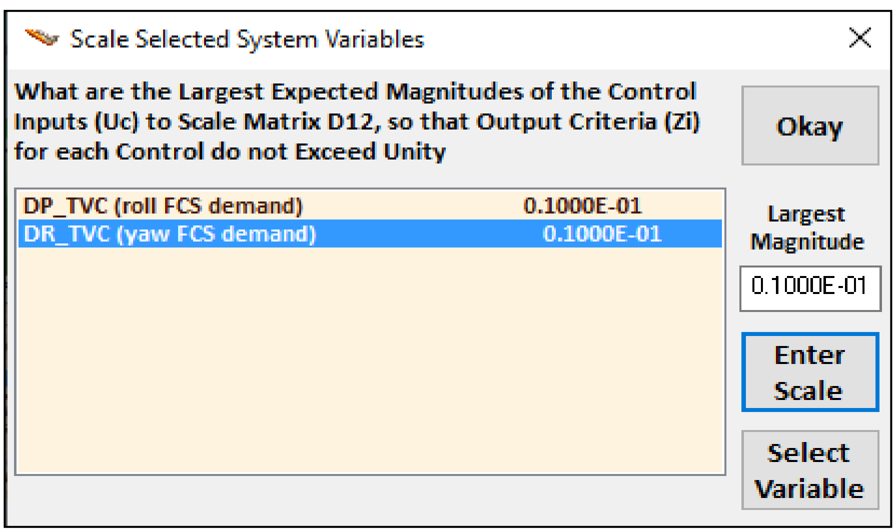


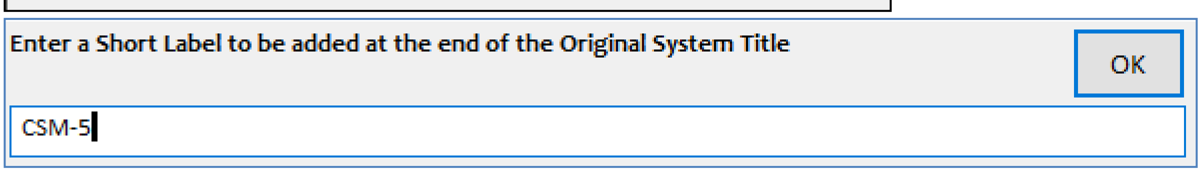


The next step is to enter gains for the performance optimization criteria. That is, the maximum acceptable magnitude at the criteria outputs defined, which are: the maximum roll and yaw attitude errors, maximum beta transient magnitude and its integral. Reducing the gain value for a specific performance output results into better performance and smaller transient for that variable. Select one variable at a time, enter the gain and click on “enter scale” to accept it. When you finish click on “Okay” to go to the next dialog.



The controls are also included in the optimization criteria. By scaling both: performance and control criteria we define the trade-off between performance, sensitivity and control bandwidth. In this example we have two controls. If we increase the gain in one of them, let’s say the roll control, we are telling the mathematic algorithm to provide more control in the roll axis which means bigger bandwidth in roll and the system will be faster in roll. Enter the two gains as before and click on “Okay” to proceed. Finally enter a short label that will appear at the end of the Synthesis Model title in the systems file.





The H-Infinity SM is saved in the systems file and it will be used to design the state-feedback controller

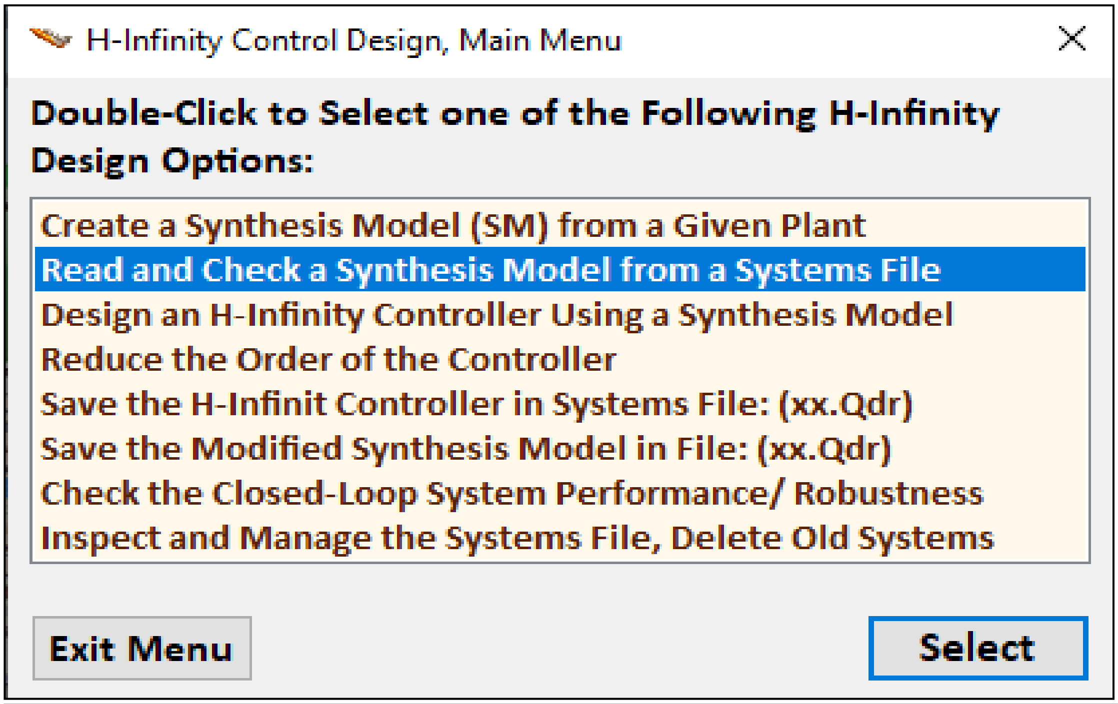
```
SYNTHESIS MODEL FOR H-INFINITY CONTROL
Shuttle Ascent, Max_Q, Design Model with TVC and Beta-Integral/SM-1
Number of: States (x), Uncertainty Inp/Outputs from Plant Variations (dP)=   6   0   0
Number of: Extern Disturbance Inputs (Wi), Control Inputs (Uc)          =   3   2
Number of: Output Criteria (Zo), Regulated Outputs (Zr), Measurements (y)=   4   0   6
Synthes Model Matrices: A, B1,B2,C1,C2, D11,D12,D21,D22, Sample Time (dT)=   0.0000
Matrix A                    Size =  6 X  6
           1-Column            2-Column            3-Column            4-Column            5-Column            6-Column
   1-Row  0.000000000000E+00  0.100000000000E+01  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00
   2-Row  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00 -0.189871516138E+01  0.000000000000E+00
   3-Row  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.100000000000E+01  0.000000000000E+00  0.000000000000E+00
   4-Row  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.113501528513E+00  0.000000000000E+00
   5-Row  0.111727524803E-01 -0.624247212969E-01  0.000000000000E+00 -0.998049689533E+00 -0.503994541829E-01  0.000000000000E+00
   6-Row  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.100000000000E+01  0.000000000000E+00
------------------------------------------------------------------------------------------------------------------------
Matrix B1                   Size =  6 X  9
           1-Column            2-Column            3-Column            4-Column            5-Column            6-Column
   1-Row  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00
   2-Row -0.159829973907E+01  0.166956228414E+00 -0.886177820482E-03  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00
   3-Row  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00
   4-Row -0.424968782616E-01 -0.107848234494E+01  0.529740001050E-04  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00
   5-Row -0.606020985044E-03  0.208540565714E-01 -0.133755484444E-04  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00
   6-Row  0.000000000000E+00  0.000000000000E+00  0.466724993040E-03  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00
           9-Column
   1-Row  0.000000000000E+00
   2-Row  0.000000000000E+00
   3-Row  0.000000000000E+00
   4-Row  0.000000000000E+00
   5-Row  0.000000000000E+00
   6-Row  0.000000000000E+00
------------------------------------------------------------------------------------------------------------------------
Matrix B2                   Size =  6 X  2
           1-Column            2-Column
   1-Row  0.000000000000E+00  0.000000000000E+00
   2-Row -0.159829973907E+01  0.166956228414E+00
   3-Row  0.000000000000E+00  0.000000000000E+00
   4-Row -0.424968782616E-01 -0.107848234494E+01
   5-Row -0.606020985044E-03  0.208540565714E-01
   6-Row  0.000000000000E+00  0.000000000000E+00
------------------------------------------------------------------------------------------------------------------------
Matrix C1                   Size =  6 X  6
           1-Column            2-Column            3-Column            4-Column            5-Column            6-Column
   1-Row  0.100000000000E+01  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00
   2-Row  0.000000000000E+00  0.000000000000E+00  0.100000000000E+01  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00
   3-Row  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.100000000000E+01  0.000000000000E+00
   4-Row  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.100000000000E+01
   5-Row  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00
   6-Row  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00
------------------------------------------------------------------------------------------------------------------------
Matrix C2                   Size =  6 X  6
           1-Column            2-Column            3-Column            4-Column            5-Column            6-Column
   1-Row  0.100000000000E+01  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00
   2-Row  0.000000000000E+00  0.100000000000E+01  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00
   3-Row  0.000000000000E+00  0.000000000000E+00  0.100000000000E+01  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00
   4-Row  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.100000000000E+01  0.000000000000E+00  0.000000000000E+00
   5-Row  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.100000000000E+01  0.000000000000E+00
   6-Row  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.100000000000E+01
------------------------------------------------------------------------------------------------------------------------
Matrix D11                  Size =  6 X  9
           1-Column            2-Column            3-Column            4-Column            5-Column            6-Column
   1-Row  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00
   2-Row  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00
   3-Row  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00
   4-Row  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00
   5-Row  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00
   6-Row  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00
           9-Column
   1-Row  0.000000000000E+00
   2-Row  0.000000000000E+00
   3-Row  0.000000000000E+00
   4-Row  0.000000000000E+00
   5-Row  0.000000000000E+00
   6-Row  0.000000000000E+00
------------------------------------------------------------------------------------------------------------------------
Matrix D12                  Size =  6 X  2
           1-Column            2-Column
   1-Row  0.000000000000E+00  0.000000000000E+00
   2-Row  0.000000000000E+00  0.000000000000E+00
   3-Row  0.000000000000E+00  0.000000000000E+00
   4-Row  0.000000000000E+00  0.000000000000E+00
   5-Row  0.100000000000E+01  0.000000000000E+00
   6-Row  0.000000000000E+00  0.100000000000E+01
------------------------------------------------------------------------------------------------------------------------
```

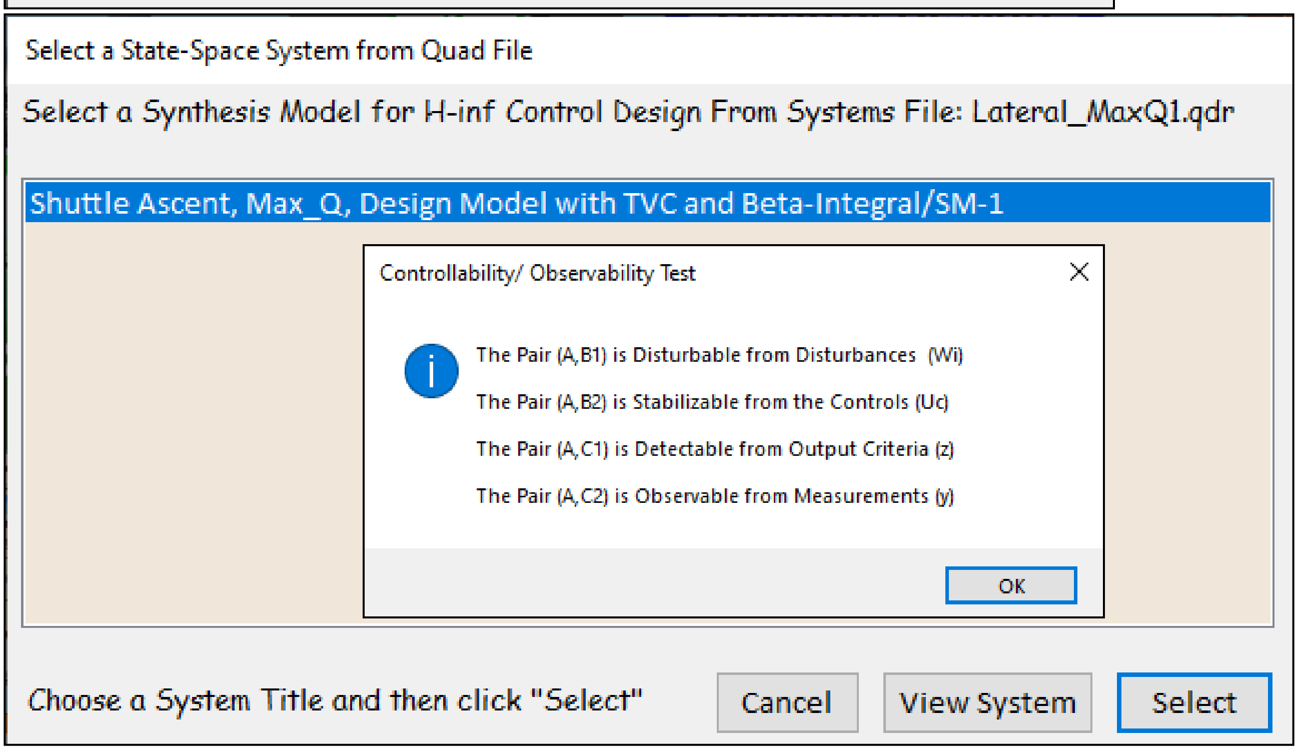
```
Matrix D21                 Size =  6 X  9
          1-Column            2-Column            3-Column            4-Column            5-Column
   1-Row  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.100000000000E+01  0.000000000000E+00
   2-Row  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.100000000000E+01
   3-Row  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00
   4-Row  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00
   5-Row  0.000000000000E+00  0.000000000000E+00  0.466724993040E-03  0.000000000000E+00  0.000000000000E+00
   6-Row  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00  0.000000000000E+00
          9-Column
   1-Row  0.000000000000E+00
   2-Row  0.000000000000E+00
   3-Row  0.000000000000E+00
   4-Row  0.000000000000E+00
   5-Row  0.000000000000E+00
   6-Row  0.100000000000E+01
----------------------------------------------------------------------------------------------------------------
Matrix D22                 Size =  6 X  2
          1-Column            2-Column
   1-Row  0.000000000000E+00  0.000000000000E+00
   2-Row  0.000000000000E+00  0.000000000000E+00
   3-Row  0.000000000000E+00  0.000000000000E+00
   4-Row  0.000000000000E+00  0.000000000000E+00
   5-Row  0.000000000000E+00  0.000000000000E+00
   6-Row  0.000000000000E+00  0.000000000000E+00
----------------------------------------------------------------------------------------------------------------
Definition of Synthesis Model Variables                       Max Scaling Factors

States (x) .........  =   6
  1    Roll Attitude   (phi-body)  (radians)
  2    Roll Rate       (p-body)    (rad/sec)
  3    Yaw  Attitude   (psi-body)  (radians)
  4    Yaw  Rate       (r-body)    (rad/sec)
  5    Angle of sideslip, beta     (radian)
  6    Beta-Integral               (rad-sec)

 Excitation Inputs (w)  =   9
   1    DP_TVC (roll FCS demand)                                    * 0.002
   2    DR_TVC (yaw FCS demand)                                     * 0.002
   3    Wind-Gust Azim, Elev Angles=(45,90)  (deg)                  * 3.0
   4    Noise at Output: Roll Attitude  (phi-body) (radians)        * 0.10000E-07
   5    Noise at Output: Roll Rate      (p-body)   (rad/sec)        * 0.10000E-07
   6    Noise at Output: Yaw  Attitude  (psi-body) (radians)        * 0.10000E-07
   7    Noise at Output: Yaw  Rate      (r-body)   (rad/sec)        * 0.10000E-07
   8    Noise at Output: Angle of sideslip, beta   (radian)         * 0.10000E-07
   9    Noise at Output: Beta-Integral             (rad-sec)        * 0.10000E-07

 Control Inputs (u)  ... =   2
   1    Control: DP_TVC (roll FCS demand)                           *  1.0000
   2    Control: DR_TVC (yaw FCS demand)                            *  1.0000

 Performance Outputs (z)=   6
   1    Roll Attitude   (phi-body)  (radians)                       / 0.0001
   2    Yaw  Attitude   (psi-body)  (radians)                       / 0.0003
   3    Angle of sideslip, beta     (radian)                        / 0.03
   4    Beta-Integral               (rad-sec)                       / 0.03
   5    Contrl Criter. DP_TVC (roll FCS demand)                     / 0.0005
   6    Contrl Criter. DR_TVC (yaw FCS demand)                      / 0.0008

 Measurement Outputs (y)=   6
   1    Measurm: Roll Attitude   (phi-body) (radians)               /  1.0000
   2    Measurm: Roll Rate       (p-body)   (rad/sec)               /  1.0000
   3    Measurm: Yaw  Attitude   (psi-body) (radians)               /  1.0000
   4    Measurm: Yaw  Rate       (r-body)   (rad/sec)               /  1.0000
   5    Measurm: Angle of sideslip, beta    (radian)                /  1.0000
   6    Measurm: Beta-Integral              (rad-sec)               /  1.0000
 -----------------------------------------------------------------------------------------------------
```

The scaling gains are included on the side of the corresponding variables to be scaled.

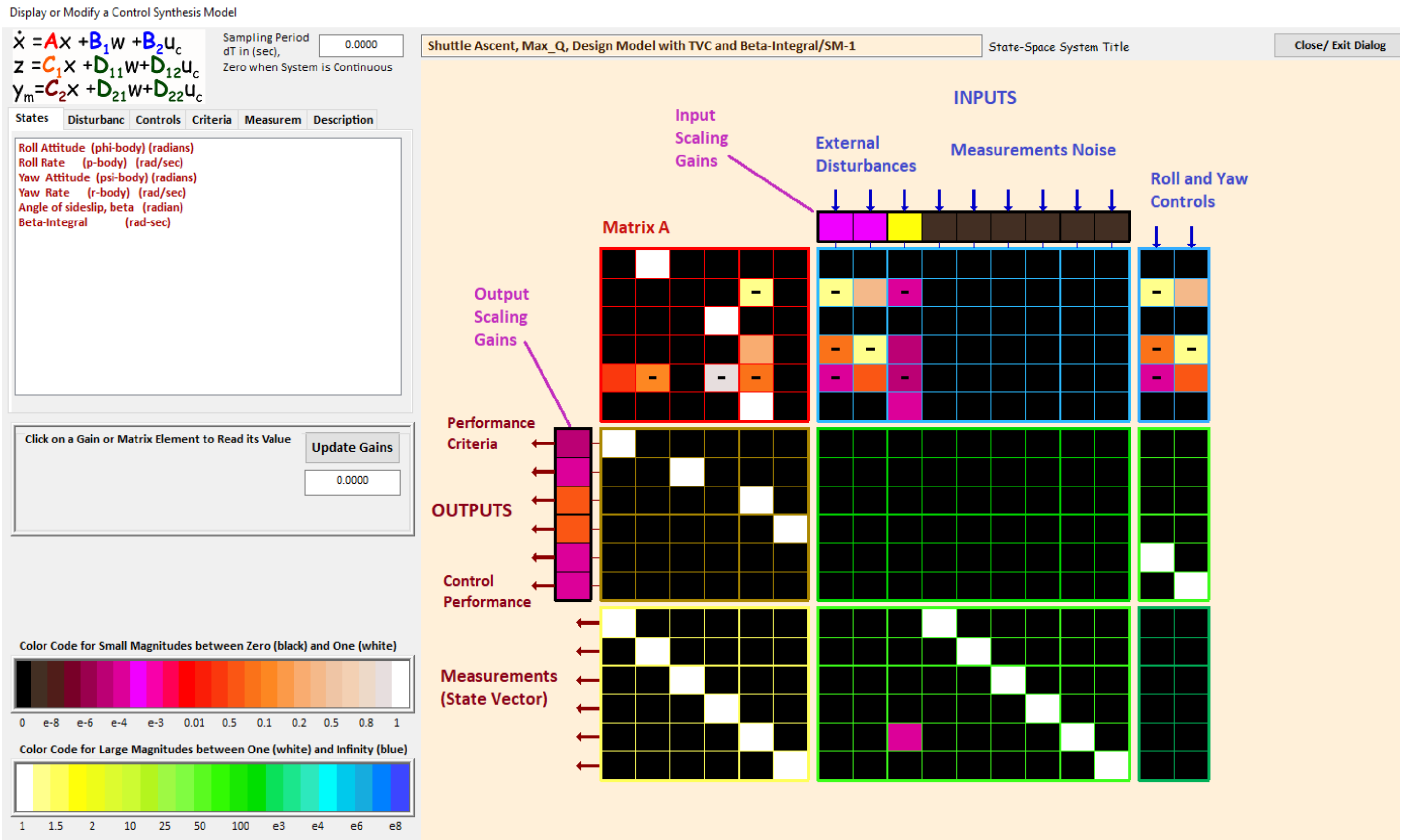
6.2 Reading and Checking the Synthesis Model

If the SM is already created and saved in the systems file, from the H-infinity main menu you choose the second option and click on "Select". The following menu shows the SM which are already saved in the systems file. In this case there is only one. Select the SM and click on "Select". The program will read the SM and check the observability and controllability.



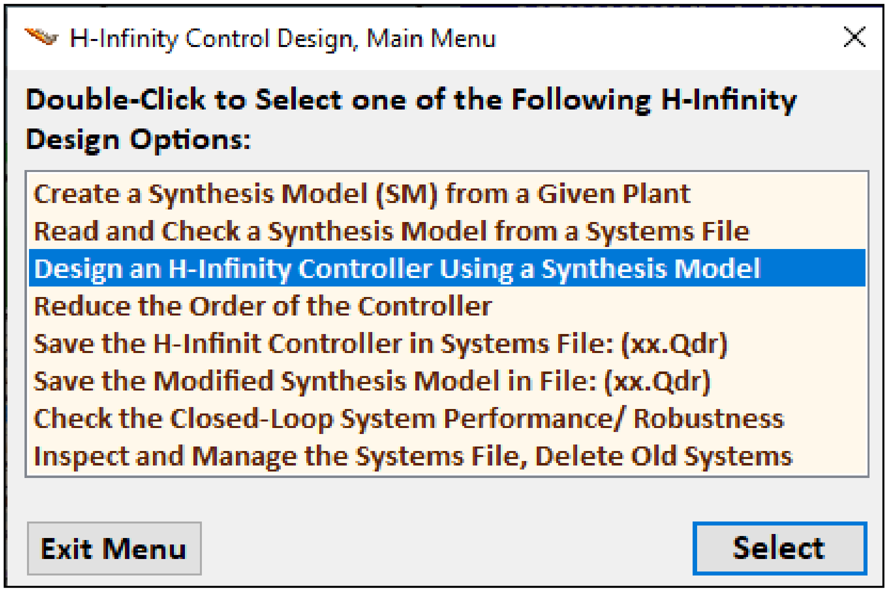


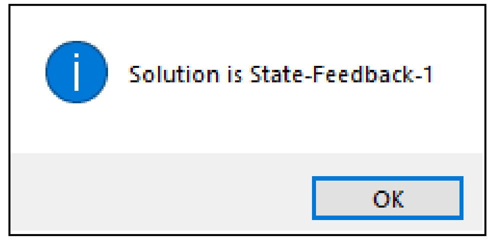
The program confirms that the SM satisfies the expected observability and controllability requirements and displays the SM matrices graphically in system's form in a dialog shown below. The 9 SM matrices appear color coded and also the gains which scale the disturbances and the criteria. The color code reference magnitudes appear at the lower-left corner. In this example the A-matrix consists of 6 states. There are 3 external disturbances, 6 measurements noise inputs which are set to almost zero (dark brown), and there are 2 control inputs for roll and yaw control. In the outputs we have 4 performance criteria, and 2 control utilization criteria. C2 is the identity matrix which means the 6 outputs are equal to the state vector. Definitions of the SM variables are listed in tabs on the left-hand side. The SM parameters can be modified interactively and the updated SM can be saved in the systems file.



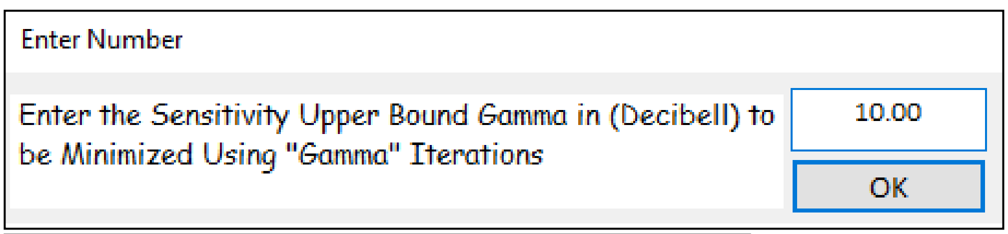
6.3 Running the H-Infinity Program Interactively

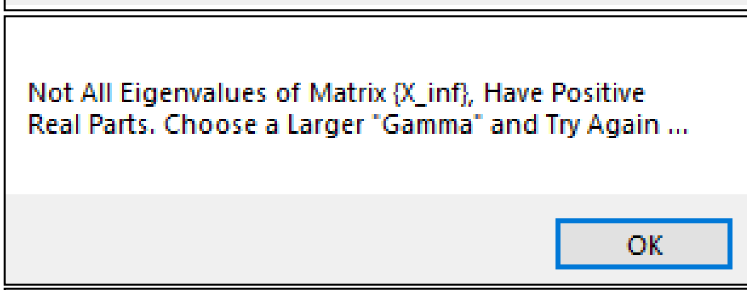
After reading and checking the SM controllability and observability you can select the third option from the main menu to design the H-infinity controller from the SM and click on "Select". In this case, the program confirms that the solution will be a state-feedback gain rather than a dynamic controller and it will use the state-feedback algorithm.

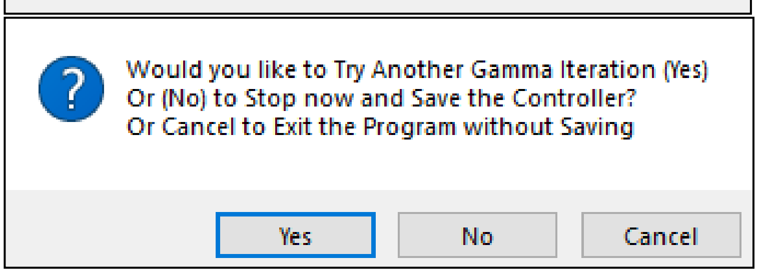


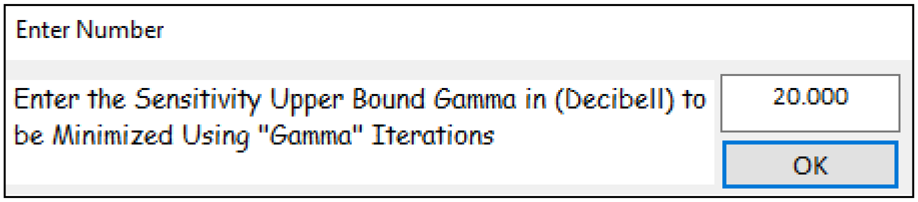


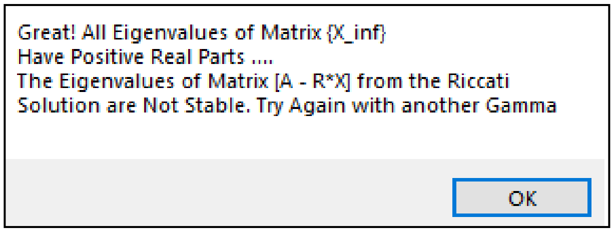
We now begin the iterative process of attempting to minimize the upper bound γ of the infinity norm of the sensitivity transfer function between the 3 disturbance inputs and the output criteria, which in this case there are 4-performance and 2-control criteria. We begin with an arbitrary γ upper bound and try to find the smallest γ magnitude in (dB) that will not violate the algorithm requirements. We must enter γ in decibels. We first enter γ=10 which is too low and click on “Yes” in the next dialog to try a bigger value. Next time we enter γ=20 which is also low and click on “Yes” again to try an even bigger value. After 2-3 iterations we find that γ=30 works and we click on “No” meaning that we do not want to try another value but to accept the current controller.

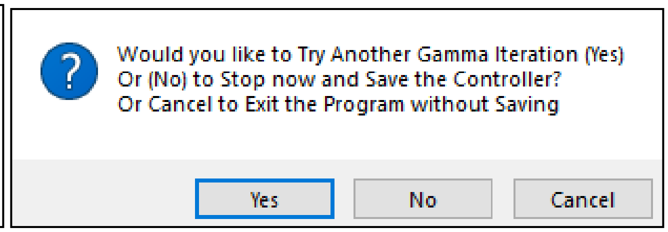




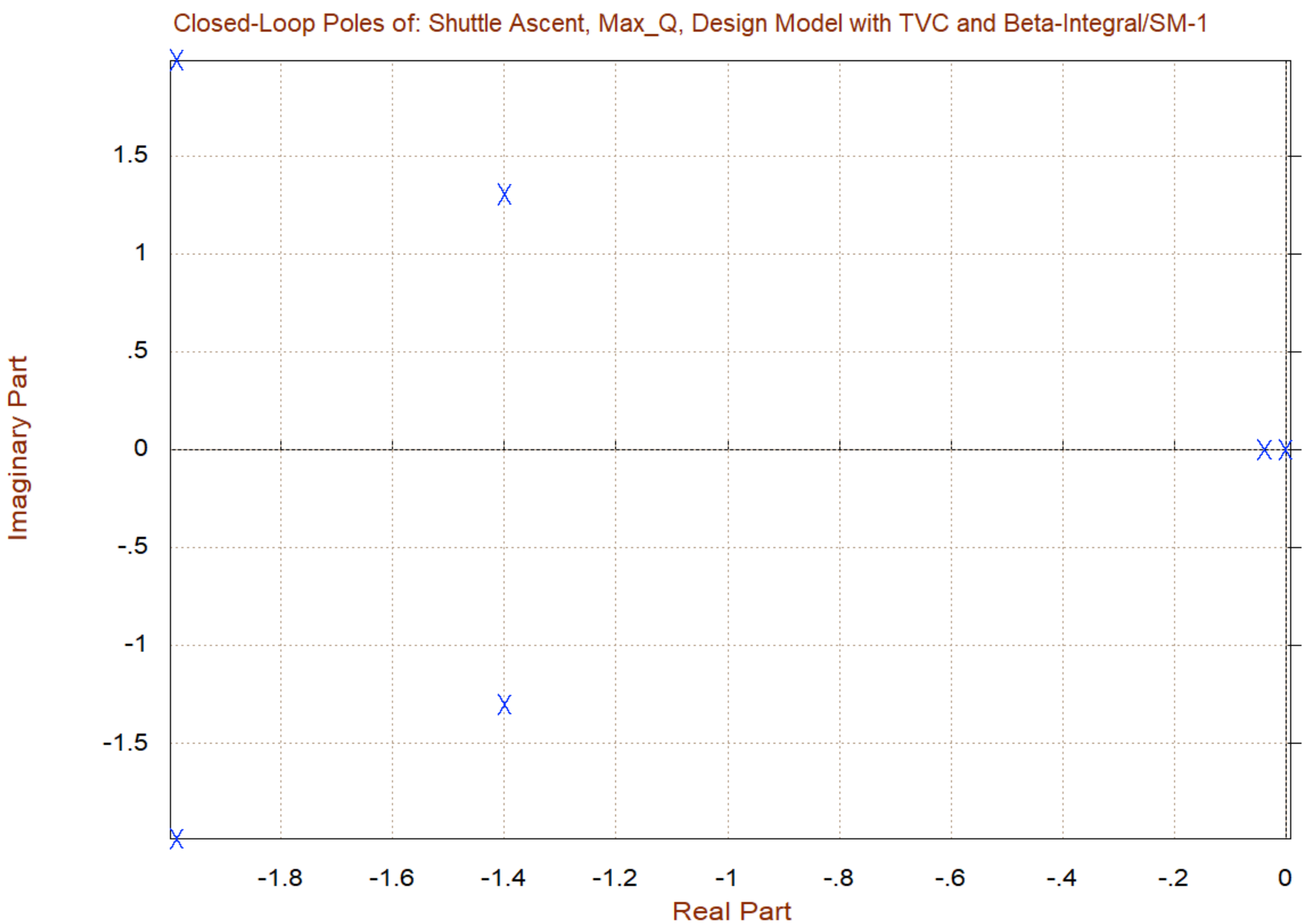








The following Figure shows the control system eigenvalues with the control loop closed between the measurements (y) and the control inputs (u). They are all stable, as expected. We return to the H-infinity main menu from where we can save the controller gain by clicking on "*Save the H-infinity Controller in Systems File (x.Qdr)*".



6.4 Running the H-Infinity Program in Batch Mode

The SM can also be created from the plant system in batch mode and the SM can be processed by the H-infinity program in batch mode to create the control system. The necessary datasets that perform those functions must be created in the input file and be processed in batch mode, either individually or via a batch dataset. In the example that follows the input file includes a dataset that creates the SM "*Crane Design Model with Y1 Integral/SM-1*" from the plant system "*Crane Design Model with Y1 Integral*". There is also a dataset that generates the controller "*H-Infin Control for Overhead Crane System*" from the SM.

```
BATCH MODE INSTRUCTIONS ...............
Batch to pdesign H-infinity controller for Overhead Crane
! Prepared the Design Model for the Overhead Crane and Performs H-infinity
! Design using Output Dynamic Feedback Control System
! and Kalman-Filter Gain and Estimator for the Overhead Crane
!
Retain System     : Overhead Crane Design Model
Transf-Function   : Integrator
System Connection: Crane Design Model with Y1 Integral
Create CSM Design: Crane Design Model with Y1 Integral/SM-1
H-Infinity Design: Overhead Crane H-Infinity Design
To Matlab Format : Overhead Crane Design Model
To Matlab Format : H-Infin Control for Overhead Crane System
-----------------------------------------------------------------------------------------------
SYSTEM OF TRANSFER FUNCTIONS ...
Integrator
INTERCONNECTION OF SYSTEMS .....
Crane Design Model with Y1 Integral
! Creates an Augmented plant for control Design by including the integral
! of mass-1 displacement in the states and output.
!
Titles of Systems to be Combined
Title 1 Overhead Crane Design Model
Title 2 Integrator
SYSTEM INPUTS TO SUBSYSTEM  1                                          Plant(s)
System Input  1 to Subsystem  1, Input  1, Gain= 1.0                   Control Force
System Input  2 to Subsystem  1, Input  2, Gain= 1.0                   Disturb Force
..............................................................................
SYSTEM OUTPUTS FROM SUBSYSTEM  1                                       Plant Outputs
System Output  1 from Subsystem  1, Output  1, Gain= 1.0               y1 displacem
System Output  2 from Subsystem  1, Output  2, Gain= 1.0               theta
System Output  3 from Subsystem  1, Output  3, Gain= 1.0               y1-dot
..............................................................................
SYSTEM OUTPUTS FROM SUBSYSTEM  2                                       Integrator
System Output  4 from Subsystem  2, Output  1, Gain= 1.0               y1 integral
..............................................................................
SUBSYSTEM NO  1 GOES TO SUBSYSTEM NO  2                                Plant Outp to Control Input
Subsystem  1, Output  1 to Subsystem  2, Input  1, Gain= 1.0           y1 displacem
..............................................................................
Definitions of Inputs  =  2
Control Force on m2    (Fc)
Disturb Force on m1    (Fd)

Definitions of States  =  5
Bottom Mass Position, y1
Top Mass Position, y2
Bottom Mass Velocity, y1-dot
Top Mass Velocity, y2-dot
Bot Mass-1 Position Integral, y1-int

Definitions of Outputs =  4
Mass-1 Displacement        (y1)
Pendulum Angle             (theta)
Bottom Mass Velocity,      (y1-dot)
Bot Mass1 Position-Integr  (y1-int)
-----------------------------------------------------------------------------------------------
```

The following dataset creates the SM. It defines which of the system inputs are controls and which are disturbances. Also, which outputs are measurements and which ones are criteria. It includes also the input and output scaling gains.

```
CREATE A SYNTHESIS MODEL FOR H-INFINITY CONTROL DESIGN
Crane Design Model with Y1 Integral/SM-1
Crane Design Model with Y1 Integral
Number of Uncertainty I/O Pairs    :    0
Number of Disturbance Inputs       :    1
Disturbance Input Numbers          :    2
Number of Control Inputs           :    1
Control Input Numbers              :    1
Number of Performance Outputs      :    4
Perform Optimization Output Numbrs:    1    2    3    4
Number of Commanded Outputs        :    1
Command Regulated Output Numbers   :    1
Number of Measurement Outputs      :    3    2
Measurement Output Numbers         :    1    2    4
Disturbance Input & Command Gains:  0.00018   0.0007    0.0012    0.0008    0.0033
Performance Output & Control Gains:  0.00015   0.5       0.2       0.0001    0.00015   0.0007
--------------------------------------------------------------------------------------------------------
H-INFINITY CONTROL DESIGN ......
Overhead Crane H-Infinity Design
Synthesis Model for Control Design in file (.Qdr) :        Crane Design Model with Y1 Integral/SM-1
Peak Value of the Sensitivity Function Gamma (dB) :        70.0
Dynamic Output Feedback via an Estimator for      :        Overhead Crane System
--------------------------------------------------------------------------------------------------------
CONVERT TO MATLAB FORMAT ........       (Title, System/Matrix, m-filename)
Overhead Crane Design Model
System
Crane
--------------------------------------------------------------------------------------------------------
CONVERT TO MATLAB FORMAT ........       (Title, System/Matrix, m-filename)
H-Infin Control for Overhead Crane System
System
Control
--------------------------------------------------------------------------------------------------------
```

The entire input file can now be processes in batch mode by running the batch set to create the Synthesis Model and the control system.